Harish Chandra Research Institute

Quantum Mechanics II

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Assignment #4

1) Compute the classical action associated with the motion of a one dimensional simple harmonic oscillator (of mass m and angular frequency ω) starting from x_i at time t_i and ending at x_f at t_f. Use this and results obtained in the lectures to obtain the explicit form of the propagator for a one dimensional simple harmonic oscillator. [10 points]

2) Show that the the propagator can be represented in terms on energy eigenfunctions as

$$K(x', t'; x, t) = \sum_n \psi_n^*(x')\psi_n(x) \text{exp}\Big[-\frac{i}{\hbar} E_n (t - t') \Big] \tag{1}$$

where $\psi_n(x)$ are the normalised wavefunctions of energy eigenstates (with energy E_n). Next, show that the partition of function of a one dimensional quantum system

$$Z = \sum_n e^{-\beta E_n}$$

can be obtained from the propagator by setting the initial and final positions to be the same ($x' = x$), making an appropriate imaginary choice for T ($T = t' - t$) which relates it to β, and carrying out an integral over x. Carry out this procedure explicitly for the simple harmonic oscillator and compare with the standard result for the partition function of the simple harmonic oscillator. Develop a similar method to compute the partition function for even states of the harmonic oscillator i.e the sum

$$\sum_n e^{-\beta E_n}$$

where the sum runs over the even whole number numbers (the energy eigenstates of the simple harmonic oscillator being labelled by whole numbers in the usual way).

[5 + 5+ 5 points]

3) Evaluate the integrals

$$\int_{-\infty}^{+\infty} dy \ e^{iy^2} \quad \text{and} \quad \int_{-\infty}^{+\infty} dy \ e^{-iy^2}$$

with the integration contour running along the real axis. Hint: You can make a variable change to bring the integrand to a form where the exponent is $-x^2$ and then suitably deform the contour. [5 points]

4) Using the results of the previous problem show that the propagator for a free particle reduces to a delta function in the limit of short time propagation. You should start from the expression for the propagator and then take a limit.

[5 points]

5) We have derived the path integral representation for potential problems in one dimension

$$H = \frac{p^2}{2m} + V(x)$$

to be

$$\langle x_f|\exp(\frac{-iH(t_f - t_i)}{\hbar})|x_i\rangle = \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{(\frac{N+1}{2})}\int\prod_k^N dx_k\,\exp\left(\frac{iS}{\hbar}\right) \quad (2)$$

where $S = S(x_f, x_N) + S(x_N, x_{N-1}) + \ldots + S(x_1, x_i)$, with

$$S(x_a, x_b) = \epsilon\left(\frac{1}{2}m\dot{x}^2 - V(x_a)\right), \quad \dot{x} \equiv \frac{(x_b - x_a)}{\epsilon}$$

Explicitly evaluate this integral for a free particle $V(x) = 0$. We carried this out for the special case $x_f = x_i = 0$ in the lectures. Perform the path integral in generality i.e with $x_f = q_f$ and $x_i = q_i$.

[10 points]

6) A non-relativistic particle of mass m in one dimension under the influence of a potential $V(x)$, $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x)$. Consider computing the propagator

$$\bar{K}(x_f, t_b; x_i, t_a) = \theta(t_b - t_a)\langle x_f, t_f|e^{-iH(t_f - t_i)/\hbar}|x_i, t_i\rangle$$

by path integral methods treating the effects of the potential perturbatively. The expression for the propagator to first order in $V(x)$ was derived in the lectures

$$\bar{K}(x_f, t_b; x_i, t_a) \approx \bar{K}_0(x_f, t_b; x_i, t_a) + \frac{i}{\hbar}\int_{-\infty}^{+\infty} dx \int_{t_a}^{t_b} dt\ \bar{K}_0(x_f, t_b; x, t)V(x)\bar{K}_0(x, t; x_a, t_a)$$

Where $\bar{K}_0$ is the propagator associated with the "free" Lagrangian $\mathcal{L} = \frac{1}{2}m\dot{x}^2$. The leading correction has a simple interpretation – sum over processes in which a free particle is affected by the presence of the potential at a single point in spacetime. Show that the second order correction is given by

$$\left(\frac{i}{\hbar}\right)^2\int_{-\infty}^{+\infty} dx \int_{t_a}^{t_b} dt \int_{-\infty}^{+\infty} dx' \int_{t_a}^{t_b} dt'\ \bar{K}_0(x_f, t_b; x', t')V(x')\bar{K}_0(x', t'; x, t)V(x)\bar{K}_0(x, t; x_a, t_a)$$

i.e sum over processes in which the interaction occurs in two space time points. What is your guess for the form and physical interpretation of the n^{th} term in the series.

[10 points]