Harish Chandra Research Institute

Quantum Mechanics II

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Assignment #5

1) In the Heisenberg picture, the evolution of operators is given by $\dot{A} = \frac{i}{\hbar}[H, A]$. Solve the above equations for the simple harmonic oscillator; express the Heisenberg picture operators for position and momentum $\big(Q_{\rm H}(t)$ and $P_{\rm H}(t)\big)$ in terms of the Schrodinger picture operators P and Q. Next, express the Heisenberg picture operators in terms of the creation and annihilation operators of the Schrodinger picture ($a^\dagger$ and a). Use this to evaluate the time ordered matrix element

$$\langle 0|T\big(Q_{\rm H}(t_1)Q_{\rm H}(t_2)\big)|0\rangle.$$

Compare with the result obtained by using path integral methods.

[3+2+3+2 points]

2) Consider a Hilbert space $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$, where $\mathcal{H}_I$ and $\mathcal{H}_{II}$ are finite dimensional of dimension m and n.

(a) Show that an arbitrary element in $\mathcal{H}$, w can be written as

$$w = \sum_{i=1}^{\rm min(m,n)} k_i u_i \otimes v_i$$

where k_i are non-negative real numbers, u_i are orthogonal vectors in $\mathcal{H}_I$, v_i orthogonal vectors in $\mathcal{H}_{II}$ and $\rm min(m, n)$ the lesser of m and n (and equal to their value if they are equal). To arrive at the above result you can use (without proving) the following property of matrices: An arbitrary $r \times s$ complex matrix T can be written as

$$T = U\Sigma V$$

where U is a $r \times r$ unitary matrix, V is $s \times s$ unitary matrix and Σ is a $r \times s$ diagonal matrix with non-negative diagonal entries (recall that a $r \times s$ diagonal matrix D has the property that all entries other than D_{ii} where $i = 1...\rm min(r, s)$ are necessarily zero).

(b) Suppose $\mathcal{H}$ describes a quantum mechanical system, w then corresponds to a state of the system. Use the result in part (a) to compute the two reduced density matrices (in terms of k_i, u_i and v_i) associated with the state w (when we divide the system into subsystems I and II) – ρ_I in which the Hilbert space II is traced out and ρ_{II} in which the Hilbert space I is traced out.

(c) Compute $S^{\rm I}_{\rm EE} = -{\rm tr}_{\rm I}\big(\rho_{\rm I}\log\rho_{\rm I}\big)$ and $S^{\rm II}_{\rm EE} = -{\rm tr}_{\rm II}\big(\rho_{\rm II}\log\rho_{\rm II}\big)$. How are they related ?

[4 + 4 + 2 points]

3) Consider a system involving two spin half particles whose translational degrees of freedom are frozen. Consider the system in the state

$$\frac{1}{\sqrt{2}}\Big(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\Big)$$

(a) Compute the reduced density matrix for the subsystem consisting of the first spin degree of freedom (obtained by tracing out the second spin degree of freedom).

(b) Compute the entanglement entropy associated with the reduced density matrix obtained in part (a).

[5 + 2 points]

4) A system T consists of two spin half particles whose translational degrees of freedom are frozen. Consider the reduced density matrix obtained by tracing out the degrees of freedom associated with the second spin. What is the maximum possible value of the associated entanglement entropy ? Is the state in T (with the same division of the system) with this amount of entanglement entropy unique ? Justify your answer.

[4 + 2 + 2 points]

5) In the statistical mechanics while working with the canonical ensemble, the probability that the system is in the state $|i\rangle$ is given by $p_i = e^{-\beta E_i}\big/Z$; where $\beta = 1\big/k_B T$, E_i the energy of the state $|i\rangle$ and Z the partition function $Z = \sum_i e^{-\beta E_i}$. The free energy is given by $F = -k_B T \ln Z$, from which one can compute the entropy $S = -\frac{\partial F}{\partial T}$. Show by making use of the above that

$$S = -k_B p_i \ln p_i$$

[10 points]

6) Consider a particle of mass m and charge e moving in a time independent magnetic field (but spatially varying) magnetic field $\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x})$. Show that the classical equations of motion can be obtained from the principle of least action with the action given by

$$S = \int dt \left(\frac{1}{2} m \dot{x}^i \dot{x}^i + eA^i(x^i(t))\dot{x}^i\right)$$

[10 points]