Harish Chandra Research Institute

Quantum Mechanics II

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Assignment #6

1) The algebra of the generators of the Lorentz group is

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\big(+ \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\sigma} M^{\mu\rho}\big). \tag{1}$$

Define the rotation and boost generations to be

$$J^i = \frac{1}{2}\epsilon^{ijk} M^{jk} \quad \text{and} \quad K^i = M^{i0}$$

By making use of (1), determine the commutators $[J^i, J^j]$, $[J^i, K^j]$ and $[K^i, K^j]$ in terms of J^i and K^i. For instance, you should find $[J^i, J^j] = i\epsilon^{ijk} J^k$.

[2 + 4 + 4 points]

2) In this problem, we will explore two dimensional representations of the Lorentz group.

(a) Consider the two dimensional matrices $M_{\rm L}^{\mu\nu}$ given by

$$M_{\rm L}^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k \quad \text{and} \quad M_{\rm L}^{0k} = -\frac{i}{2}\sigma^k,$$

where σ^k are the Pauli matrices. Show by explicit computation that the matrices furnish a representation of the algebra Lorentz generators. Hint: The results of problem 1 can be useful.

(b) Consider a second set of two dimensional matrices $M_{\rm R}^{\mu\nu}$ given by

$$M_{\rm R}^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k \quad \text{and} \quad M_{\rm R}^{0k} = +\frac{i}{2}\sigma^k.$$

Show by explicit computation that these matrices also furnish a two dimensional representation of the algebra of Lorentz generators.

(c) Let ψ be a two component field which transforms under Lorentz transformations in the representation given in part (a). Show that $\chi \equiv \sigma^2\psi^*$ (where σ^2 is the second Pauli matrix) transforms under the representation given in part (b). Hint: You might like to focus on infinitesimal transformations and make use of the identity $\sigma^2\big(\sigma^i\big)^* = -\sigma^i\sigma^2$.

[5 + 3 + 7 points]

3) Show that under a boost along the axis pointing along the unit vector n^j, a Dirac spinor tranforms as

$$\psi(x^\nu) \rightarrow \exp\Bigg(+ iS^{j0} n^j \beta\Bigg)\psi\big((\Lambda^{-1})^\nu{}_\mu x^\mu)\big),$$

where $\Lambda^\alpha{}_\beta$ is the 4×4 matrix which gives the action of the boost on four vectors, $S^{j0} = \frac{i}{2}\gamma^j\gamma^0$ and β the rapidity.

[8 points]

4) Let $\psi(x)$ be a Dirac spinor which satisfies the Dirac equation. Define the four current $J^\mu(x)$ by $J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$. Show that the current is conserved i.e. $\partial_\mu J^\mu = 0$.

[8 points]

5) By taking a Dirac spinor $\psi(x)$ in the plane wave ansatz with positive energy

$$\psi(x) = u(p)e^{-ip_\mu x^\mu}, \quad \text{with} \quad p^0 = \sqrt{|\vec{p}\,|^2 + m^2}$$

and applying the Dirac equation we obtained

$$u(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\xi \\ \sqrt{p_\mu \bar{\sigma}^\mu}\xi \end{pmatrix}$$

where $\sigma^\mu = (1, \vec{\sigma})$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and ξ a constant (with two components). By carrying out a similar analysis obtain the plane wave like negative energy solutions of the Dirac equation.

[9 points]

6) (a) Consider a ball of radius R which has uniform charge density and uniform mass density rotating about an axis with constant angular velocity in the presence of a constant magnetic field $\vec{B}$. Compute the energy of the configuration as a function of the angle between the axis of rotation and the magnetic field. Determine the gyromagnetic ratio of the ball. Compare with the gyromagnetic ratio obtained for the electron from Dirac theory.

(b) The non-relativistic Hamiltonian for an electron in the presence of a constant magnetic field and vanishing electric field (in a gauge in which the vector potential is taken to be time independent) is given by

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 - \frac{e}{mc}\vec{S}.\vec{B}$$

where $\vec{S}$ is the spin operator. Take the magnetic field to be along the z-axis; $\vec{B} = B_0\hat{z}$. Working in the gauge in which $\vec{\nabla}.\vec{A} = 0$ and to linear order in B_0, deduce the coefficient of the $\vec{L}.\vec{B}$ coupling (where $\vec{L}$ is the orbital angular momentum). What is the associated gyromagnetic ratio ?

[5 + 5 points]

7) Obtain the equations of motion of a Dirac spinor in the presence of an external time-independent electric field. Take the non-relativistic limit of the equation, compare your result with the Schrodinger equation describing the motion of an electron in the presence of an electric field.

[8 points]