Should I really reconstruct a state from that data?

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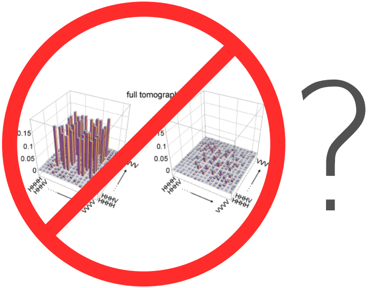

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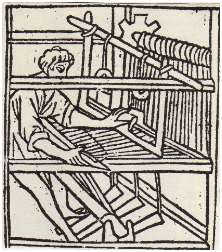
Tomography: issues, tweaks & tricks

Reasons for state tomography:

- demonstration of a good control of a quantum system
- experimental verification of interesting states
- most general dataset for later analysis



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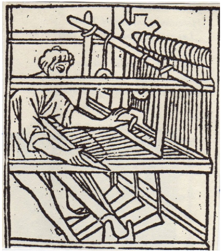

Difficulties:

- exponential cost in measurement time
  $\hookrightarrow$ compressed sensing, permutation invariant tomography, ...
- exponential cost of classical computation time
  $\hookrightarrow$ compressed sensing, permutation invariant tomography, ...
- low sampled data with low statistical significance
  $\hookrightarrow$ maximum likelihood estimation, Bayesian methods, maximum entropy principle, Ockham's razor...

Why maximum likelihood?

- naïve state reconstruction does not yield a valid quantum state (negative eigenvalues)
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> The maximum likelihood methods yields a valid quantum state, even for completely messed up data.

- Why do we get an nonphysical state in the first place?
- What if systematic errors are present?

Simple schemes

Example: for Q qubits, measure locally all Pauli operators,

$$\begin{aligned} \alpha = (z, z, \dots): &\quad \sigma_z \otimes \sigma_z \otimes \cdots , \\ \alpha = (z, x, \dots): &\quad \sigma_z \otimes \sigma_x \otimes \cdots , \\ \alpha = (x, z, \dots): &\quad \sigma_x \otimes \sigma_z \otimes \cdots \\ &\dots \end{aligned}$$

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More generally:

Experimenter performs measurements $E_{i|\alpha}$ yielding

$$p_{i|\alpha} = \mathrm{tr}(E_{i|\alpha}\, \varrho_{\mathrm{exp}}) \text{ or } \mathbf{p} = M[\varrho_{\mathrm{exp}}].$$

Complete tomography, if $M[\varrho] \neq M[\varrho']$.

Low sampling issues

- in an N_α-fold measurement of setting α, we sample $(n_{\alpha,1}, n_{\alpha,2}, \dots)$ from the multinomial distribution $(p_{\alpha,1}, p_{\alpha,2}, \dots)$

↪ frequencies $f_{i|\alpha} = n_{i|\alpha}/N_\alpha$

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Innocent state reconstruction

Least square $\sum_{i|\alpha}(f_{i|\alpha} - M[\varrho])^2$ yields $\varrho_{\rm ls} = M^{-1}[\,\mathbf{f}\,]$.

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Is ϱ_{ls} a bad idea?

$+$ easy and fast to compute

$+$ converges to true value

$-$ not appropriate for small N_α

$-$ has negative eigenvalues

Hoeffding beats negative eigenvalues

Suppressed negative expectation values

Choose an arbitrary vector $|\psi\rangle$. Then for $t > 0$,

$$\Pr[\,\langle\psi|\varrho_{\mathrm{ls}}|\psi\rangle < -t\,] \leq \exp[-t^2 N/\mathrm{const}_\psi].$$

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A test procedure:

- take two datasets I and II for the same ϱ_{exp}
- choose $|\psi\rangle$ according to $\varrho_{\mathrm{ls}}^{\mathsf{I}}$
- then it is unlikely to observe $\langle\psi|\varrho_{\mathrm{ls}}^{\mathsf{II}}|\psi\rangle < -t$

$\hookrightarrow$ negative eigenvalues are in random directions

The issue of overcomplete tomography

- Usually tomography is performed in an overcomplete setup.
- Example: 3^Q Pauli measurements with $2^Q - 1$ outcomes vs. $4^Q - 1$ entries in the density matrix.
- Reduction from $\sim 6^Q$ to $\sim 4^Q$ dimensions.

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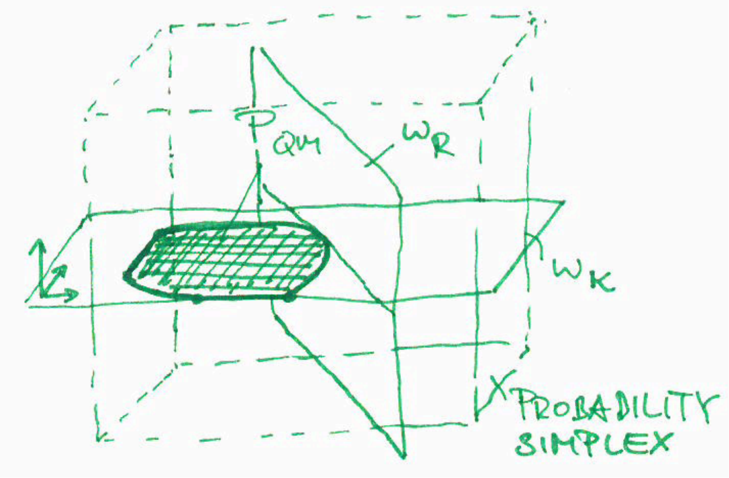
Linear dependencies

Let $\mathbf{p}_{\mathrm{ls}}$ be the probabilities predicted from ϱ_{ls}. Then for $t > 0$,

$$\Pr[\,|(\mathbf{p}_{\mathrm{ls}}^{\mathsf{I}} - \mathbf{f}^{\mathsf{I}}) \cdot (\mathbf{p}_{\mathrm{ls}}^{\mathsf{II}} - \mathbf{f}^{\mathsf{II}})| > t\,] \leq 2\exp[-t^2 N/\mathrm{const}_{\mathbf{f}^{\mathsf{I}}}].$$

$\hookrightarrow$ *Deviation from predicted probabilities are not systematic.*

Like a witness



We entered the regime of hypothesis testing

Under the assumption (data) $\sim \mathbf{p} = M[\varrho_{\mathrm{exp}}]$, we arrived at

$$\Pr[\,T(\text{data}) > t\,] \leq \exp[-t^2 N/\mathrm{const}_T]$$

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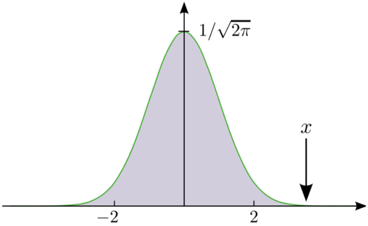
Sample from Gaussian process, get value x, then

$$\Pr[|x| > t] = 1 - \int_{-t}^{t} \mathrm{e}^{-x^2/2}/\sqrt{2\pi}\,\mathrm{d}x.$$



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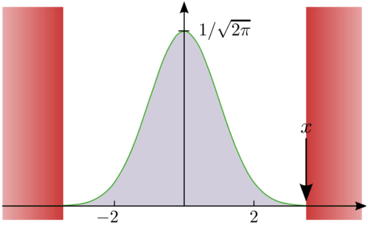

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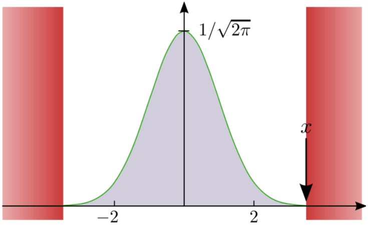

- We say, x is *excluded* by a x-σ region,

e.g. 1σ: 31.7%, 2σ: 4.56%, 3σ: 0.270%, ...

- Let's apply this to our methods – shortly.

Likelihood ratio test

- In overcomplete tomography, M fails to be surjective ($6^Q \longrightarrow 4^Q$).

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Likelihood ratio test/Wilks theorem

If $\mathbf{p} \in \operatorname{range} M$, then, as $N \to \infty$,

$$\Pr[\,2N \inf\{D(\mathbf{f}\|\mathbf{p}) \mid \mathbf{p} \in \operatorname{range} M\} < t] \to Q(\Delta/2, t/2)$$

where

- $D(\mathbf{f}\|\mathbf{p}) = \mathbf{f} \cdot \log \mathbf{f} - \mathbf{f} \cdot \log \mathbf{p}$ is the relative entropy,
- $Q(s,x) = \Gamma(s,x)/\Gamma(s)$ is the regularized Gamma function,
- $\Delta = \dim \operatorname{range}(M)^\perp$ is the dimension deficit.

Empirical results

Given the data, can we exclude, that $\mathbf{p} = M[\varrho_{\text{exp}}]$ is a valid model?

state	Q	N	X-talk	w_K	w_R	LR	LR*
GHZ	4	2500	20%	4.0σ	14σ	19σ	$>3.3\sigma$
		750	12%	–	5.0σ	3.6σ	3.3σ
		300	$<3\%$	0.3σ	0.7σ	(2.6σ)	2.0σ
Bell	2	61650	$<3\%$	–	–	0.6σ	0.7σ
$\lvert\uparrow\uparrow\uparrow\uparrow\rangle$	4	250	?	1.6σ	0σ	(3.4σ)	2.8σ
BE	4	5200	$<3\%$	0.08σ	0.8σ	0.9σ	0.9σ
W	5	100	4%	0.6σ	0.1σ	(3.3σ)	1.8σ

Summary

- The negative eigenvalues in a linear reconstruction yield very unlikely a negative expectation value.
- Systematic errors can be distinguished from statistical errors using witness-like structures or the likelihood ratio test.
- Our method works on current experimental data.
- [arXiv:very.soon]