Extending the domain of validity of Lagrangian Perturbation Theory (LPT)

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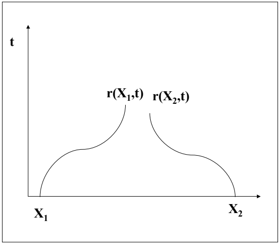
Motivation: Need for analytical modeling of the non-linear regime.

- Growth of large scale structure has emerged as a powerful probe of properties of the universe.
- Small fluctuations are well modeled by linear theory. Gravity is non-linear. Higher density contrasts are typically modeled by simulations
- Numerical simulations are:
  - Slow. Expensive to generate mock catalogues.
  - Discrete particle representation gives rise to shot noise. Higher redshift $\implies$ higher resolution.
    Simulations with NG initial conditions:
    $256^3 - 512^3$ particles for a redshift of $z \sim 49$, (Wagner *et al* 2010, Dalal *et al* 2007).
    Sefusatti *et al* (2010) used 1024^3 to start at $z \sim 99$.
- Analytical techniques can be faster and deal with smooth fields so they are not shot-noise limited.

Lagrangian theory

- Lagrangian Framework
  (Zeldovich 1970, Moutarde *et al* '91; Bouchet *et al* '92,'95; Buchert '92, '94; Buchert & Elhers '93, '96.)
  - Position is the main variable expressed as functions of initial particle labels **X** and time t.
  - The acceleration $\ddot{\mathbf{r}}$ is given by Newton's law of gravity.
  - But denisty is re-constructed from its exact non-perturbative definition.

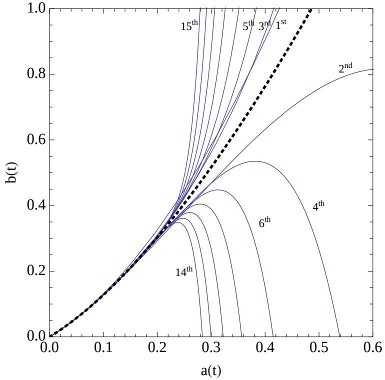
$$
\begin{aligned}
\mathbf{r} = \mathbf{r}(\mathbf{X}, t) &= a(t)\mathbf{X} + \mathbf{p}(\mathbf{X}, t) \\
\rho_m(\mathbf{X}, t) &= \frac{\rho_m(\mathbf{X}, t_0) J(\mathbf{X}, t_0)}{J(\mathbf{X}, t)} \\
J(\mathbf{X}, t) &= Det\left(\frac{\partial r_i}{\partial X_j}\right)
\end{aligned}
$$



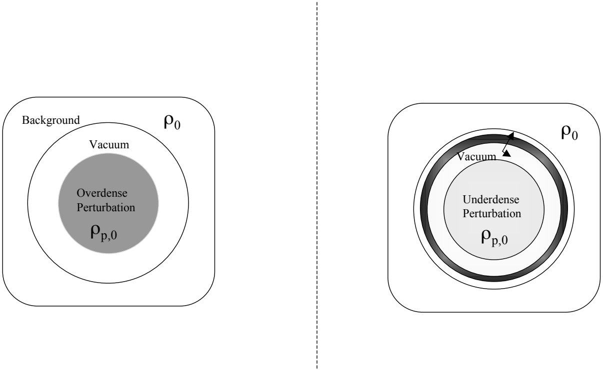
The framework is valid until orbit-crossing, beyond which one needs to invoke some sort of pressure effects.

However, LPT does not always converge.

- Sahni & Shandarin (MNRAS 1996) showed that the series does not converge for spherical homogenous voids. $b(t)$ is the scale factor of the edge of the homogenous void.
- Tatekawa (PRD 2007) attempted to correct this using the Shanks transformation. However, such nonlinear transformations of the series are not guaranteed to work.
- Aim: To understand and correct for this problem.



The compensated spherical-top hat: set up



$$\delta = \frac{\rho_{p,0}}{\rho_0} - 1$$

$$\delta_v = \frac{H_{p,0}}{H_0} - 1$$

The compensated spherical-top hat

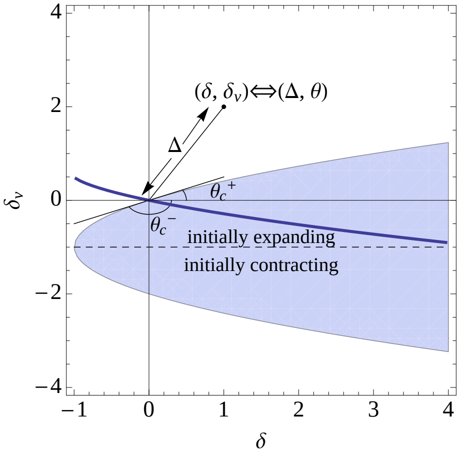
- Two independent initial conditions: Initial density δ and initial velocity δ_v.

- Total energy determines whether model can be open or closed.

- In general, background and perturbation have different big bang times. Imposing equality gives blue curve - Zeldovich curve.

- Scale factor of the system is written as $b(\Delta, t) = \sum_{n=0}^{\infty} b^{(n)}(t)\Delta^n$

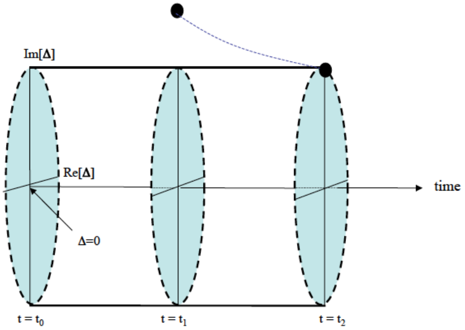
$$\begin{aligned} \Delta &= \sqrt{\delta^2 + \delta_v^2} \\ \delta &= \Delta \cos\theta \\ \delta_v &= \Delta \sin\theta \end{aligned}$$



Complex analysis of the exact solution

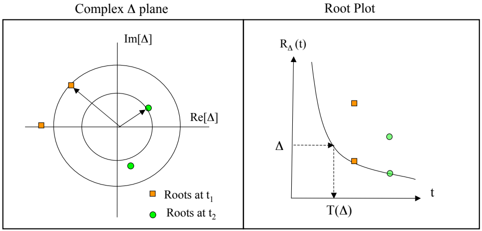
- Allow Δ to be complex. $\Delta \to \boldsymbol{\Delta}$. $\mathbf{b}(\boldsymbol{\Delta}, t)$ is complex. Locate singularities of $\mathbf{b}(\boldsymbol{\Delta}, t)$ in the complex $\boldsymbol{\Delta}$ plane.
- For this system the condition $\mathbf{b}(\boldsymbol{\Delta}, t) = 0$ gives the condition of the pole.

Disk with radius equal to initial $\Delta \longrightarrow$

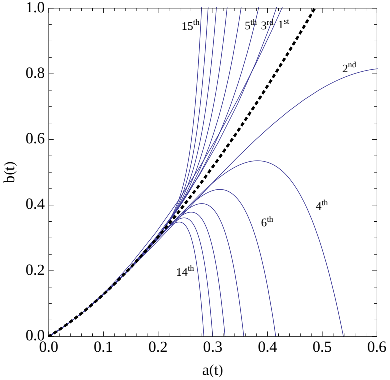


Radius of convergence vs. Time of validity

- Given an initial Δ, the time of validity is the time when a singularity first occurs within a disk of radius Δ in the complex $\boldsymbol{\Delta}$ plane. It determines how long the series solution is valid.
- Given a fixed time interval, the radius of convergence determines the range of initial amplitudes that will converge over that time interval.
- The roots can be real or complex. Real roots of a closed model correspond to 'collapse'.



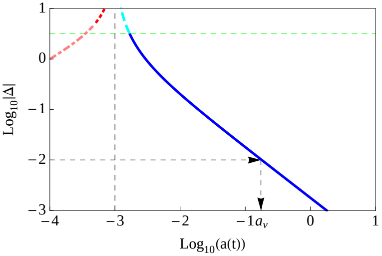
Results I: Open models explained



Model parameters

$\Delta = 0.01$, $\theta = 2.82$.

ROOT PLOT for $\theta = 2.82$

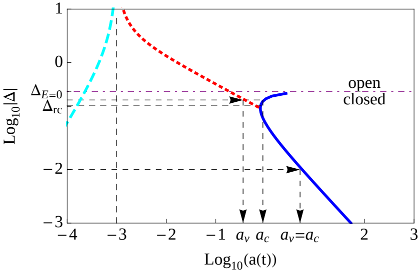


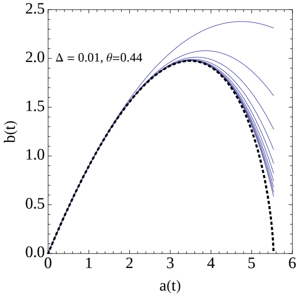
Read off the time of validity:
$a_{valid} = 0.179$

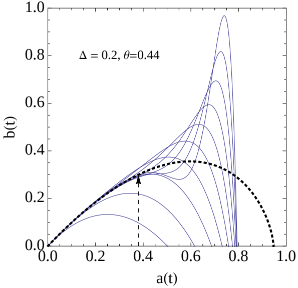
Results II: Convergence for closed models not guaranteed

It was believed that for closed models LPT always converges until the model collapses. However, we find that for some range of phase space this is not guaranteed.

Root plot for $\theta = 0.44$.

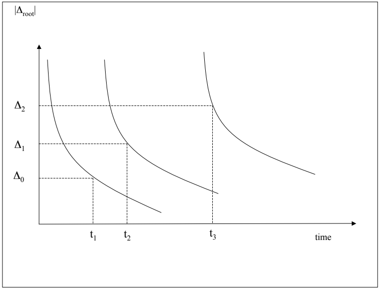






Fix the problem: LPT re-expansion

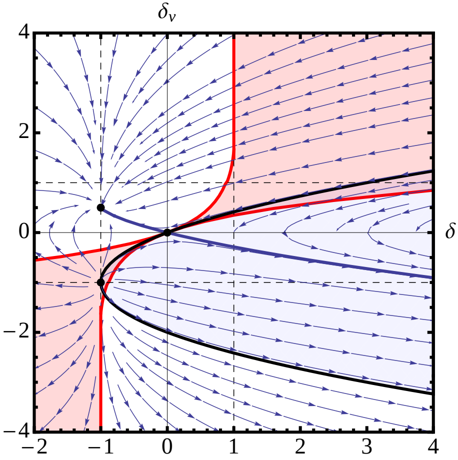
- Given an initial Δ and θ, propagate the series for a time less than the time of validity.
- Re-evaluate Δ and θ and propagate the series until a time less than the new time of valdity.



Does this work and is this feasible ?

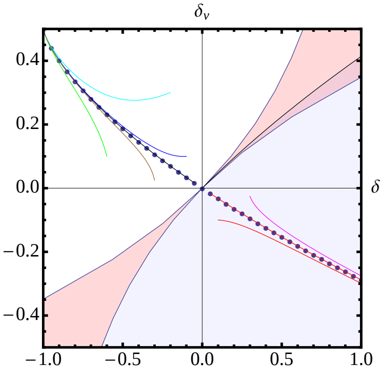
Dynamics in Phase Space

- Red shaded region denotes areas where complex roots limit convergence.
- Blue line is the equal big bang time ‘Zeldovich curve’.
- Open models move towards the attracting fixed point at $(-1, 0.5)$. Closed models move parallel to the Zeldovich line out toward ∞.
- LPT re-expansion amounts to moving along the lines in a discrete manner.
- Closed models evolve out of the discrete region i.e LPT re-expansion in necessary for convergence.



Cosmological initial conditions start near the origin and usually along the Zeldovich line.

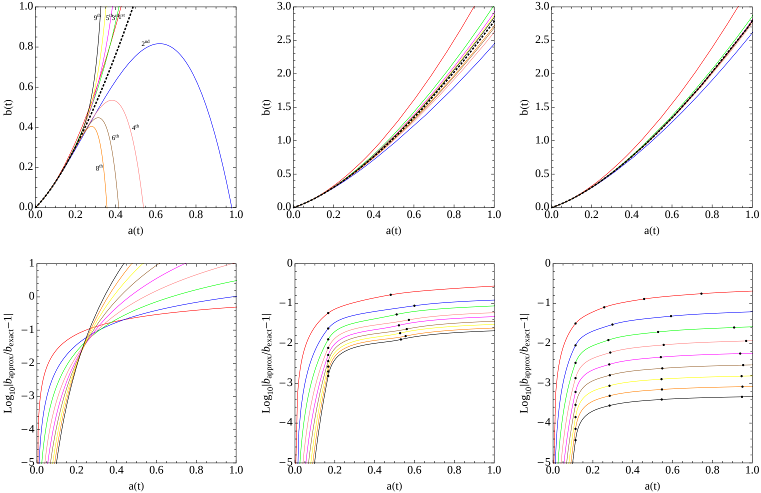
Feasibility



$T_{valid} \propto \frac{f(\delta,\delta_v)}{H(t)}$, where $H(t)$ is the Hubble parameter of the background. δ, δ_v approach a const. for open models.

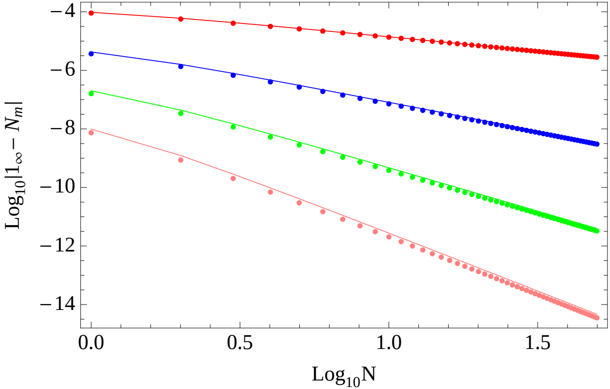
$H(t)$ decreases, therefore time of validity always decreases. δ decreases for closed models in shaded region, but $f \sim \Delta^{-2.5}$.

Demonstration of the scheme: Open models



One step | Three steps | Five steps

Convergence rate



Small total time step $(t_f - t_i)/t_i << 1$. Lines are different Lagrangian orders.

- Better convergence is achieved by increasing Lagrangian order and/or frequency of re-expansion.
- Even a first order scheme with multiple steps is capable of achieving the right answer.
- Higher the order better is the convergence rate.

Conclusion

- The convergence properties of LPT were examined from first principles and the conclusion is that for LPT to be a stable scheme, like any finite difference scheme, it should be applied iteratively.
- Such schemes do not suffer from shot-noise effects and can be started at any redshift.
- Can be fine tuned to achieve any desired accuracy by changing Lagrangian order and number of steps, modulo finite grid size effects.
- The method of Lagrangian re-expansions addresses issues of convergence *before shell-crossing*. LPT remains limited to scales that have not undergone shell crossing.
- To model scales which are highly non-linear today it is necessary to include multi-streaming effects, which may involve adding a pressure term.