Excursion Sets and Primordial Non-Gaussianity

Aseem Paranjape

The Abdus Salam ICTP, Trieste

in coll. with G. D'Amico, M. Musso and J. Noreña

1005.1203 (JCAP in press)
1011.1229 (PRD submitted)

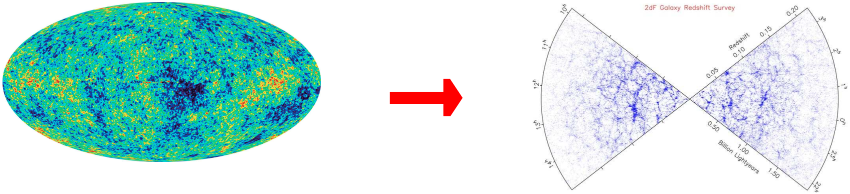
HRI, Allahabad, December 2010

Plan

1. Introduction
2. Mass function for halos
   - MR prescription
   - Improving the mass function
3. Mass function for voids
   - Two barriers and “voids in clouds”

Introduction

Primordial fluctuations grow under gravity to form large scale structure today.



Primordial *statistics* are therefore imprinted in statistics of LSS.

Excursion set framework provides an analytical mapping between the two. Focus here is on the halo mass function, i.e. – the mass distribution of virialized dark matter halos. (Later also the void mass function.)

Introduction
Excursion set formalism

Two main ingredients :

Spherical collapse

A spherical region of initial (Lagrangian) radius R and overdensity $\delta_{R,i}$ will expand, turn around and eventually "collapse".
(Ideally – to a point; realistically – to a virialized object.)

Collapse will occur today if $\delta_{R,i} = a_i\, \delta_{\rm c}$ where $\delta_{\rm c} \simeq 1.686$.

Convenient to use "linearly extrapolated" density $\delta_R \equiv \delta_{R,i}/a_i$.

Introduction
Excursion set formalism

Two main ingredients :

Spherical collapse

A spherical region of initial (Lagrangian) radius R and overdensity $\delta_{R,i}$ will expand, turn around and eventually "collapse".
(Ideally – to a point; realistically – to a virialized object.)

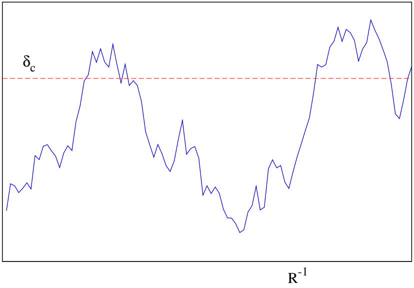
Collapse will occur today if $\delta_{R,i} = a_i\,\delta_{\rm c}$ where $\delta_{\rm c} \simeq 1.686$.

Convenient to use "linearly extrapolated" density $\delta_R \equiv \delta_{R,i}/a_i$.

Random walks and first-passage

Assume that filtered, lin^{ly} extr^{d} density contrast at $\vec{x} = 0$:
$\hat{\delta}_R = (2\pi)^{-3}\int d^3k\widetilde{W}(kR)\hat{\delta}(\vec{k})$
plays the role of δ_R.

Introduction
Excursion set formalism

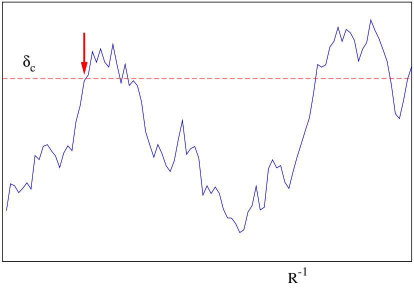


Random walks and first-passage

Assume that filtered, lin^{ly} extr^{d} density contrast at $\vec{x} = 0$:
$\hat{\delta}_R = (2\pi)^{-3} \int d^3k \widetilde{W}(kR)\hat{\delta}(\vec{k})$
plays the role of δ_R.

As Lagrangian radius R is decreased, $\hat{\delta}_R$ performs a random walk. If the walk crossed δ_c on a scale R_*, assume that an object of mass $M_* \propto R_*^3$ forms today.

Introduction
Excursion set formalism



Random walks and first-passage

Assume that filtered, lin[ly] extr[d] density contrast at $\vec{x} = 0$:
$\hat{\delta}_R = (2\pi)^{-3} \int d^3k \widetilde{W}(kR) \hat{\delta}(\vec{k})$
plays the role of δ_R.

As Lagrangian radius R is decreased, $\hat{\delta}_R$ performs a random walk. If the walk crossed δ_c on a scale R_*, assume that an object of mass $M_* \propto R_*^3$ forms today.

We look for the largest scale on which δ_c is crossed, since physically this object will crush any smaller overdense regions. Hence we want the first passage of the barrier δ_c as R is decreased.

Introduction
Excursion set formalism

Excursion set logic :

- Variance $S \equiv \sigma_R^2 = \langle \hat{\delta}_R^2 \rangle = (2\pi)^{-3} \int d^3k \widetilde{W}(kR)^2 P_\delta(k)$ plays the role of a natural "time" for the random walk. As $R \to \infty$, $S \to 0$.
- Details of the stochastic process depend on choice of filter. For the sharp-k filter and Gaussian initial conditions, the process is Markovian.
- With barrier at δ_c, consider distribution $\mathcal{F}$ of "first-crossing times" for ensemble of random walks.

Introduction
Excursion set formalism

Excursion set logic :

- Variance $S \equiv \sigma_R^2 = \langle \hat{\delta}_R^2 \rangle = (2\pi)^{-3} \int d^3k \widetilde{W}(kR)^2 P_\delta(k)$

  plays the role of a natural "time" for the random walk. As $R \to \infty$, $S \to 0$.
- Details of the stochastic process depend on choice of filter. For the sharp-k filter and Gaussian initial conditions, the process is Markovian.
- With barrier at $\delta_{\rm c}$, consider distribution $\mathcal{F}$ of "first-crossing times" for ensemble of random walks.
- $\mathcal{F}$ can be related to observable number density of collapsed objects of mass $M = (4\pi/3)\bar{\rho}R^3$:

$$\frac{dn}{dM} = \frac{\bar{\rho}}{2M^2} f(S) \left| \frac{d \ln S}{d \ln M} \right| \quad ; \quad f(S) \equiv 2S\mathcal{F}(S) \, .$$

We refer to $f(S)$ as the "mass function" (usually called multiplicity). Throughout, "time" refers to the variance of density fluctuations $S = \sigma_R^2 = \sigma^2(M)$. Bkgnd cosmology is WMAP7-compatible ΛCDM.
The result for Gaussian init. condns. and the sharp-k filter is (Bond *et al.* (1991))

$$f_{\rm PS}(\nu) = \sqrt{\frac{2}{\pi}}\, \nu e^{-\frac{1}{2}\nu^2} \quad ; \quad \nu = \delta_{\rm c}/\sigma_R \, .$$

Introduction
Primordial non-Gaussianities (NG)

Characteristic imprint on LSS is via non-vanishing connected moments (cumulants) of $\hat{\delta}_R$, e.g. $\langle\, \hat{\delta}_{R_1} \hat{\delta}_{R_2} \hat{\delta}_{R_3} \,\rangle_c$. The simplest of these are the "equal-time" cumulants, which we parametrize as

$$\varepsilon_1 \equiv \frac{\langle\, \hat{\delta}_R^3 \,\rangle_c}{\sigma_R^3} \quad ; \quad \varepsilon_2 \equiv \frac{\langle\, \hat{\delta}_R^4 \,\rangle_c}{\sigma_R^4} \,,$$

and so on. In particular, $\varepsilon_1 = \sigma \mathcal{S}_3$, $\varepsilon_2 = \sigma^2 \mathcal{S}_4$, etc. where $\mathcal{S}_3$, $\mathcal{S}_4$, etc. are reduced cumulants.

For primordial NG, the ε_n remain approximately constant on scales of interest. E.g., for local model with $f_{\rm NL}^{loc} = 100$, $\varepsilon_1 \simeq 0.02$.

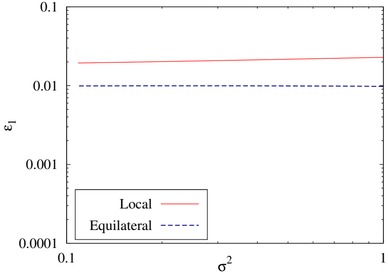
Primordial curvature perturbation : $\mathcal{R}(\vec{x}) = \mathcal{R}_g(\vec{x}) + \frac{3}{5} f_{\rm NL}^{\rm loc} \left(\mathcal{R}_g^2(\vec{x}) - \langle\, \mathcal{R}_g^2 \,\rangle \right) + \frac{9}{25} g_{\rm NL} \mathcal{R}_g^3(\vec{x})$

Sub-horizon Bardeen potential : $\Phi(\vec{k}, z) = -\frac{3}{5} T(k) \frac{D(z)}{a} \mathcal{R}(k)$

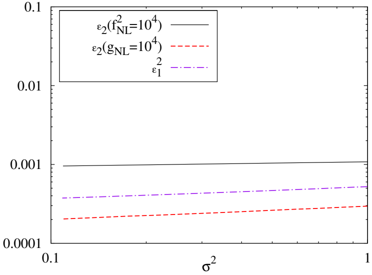
Density contrast : $\delta(\vec{k}, z) = -\frac{2ak^2}{3\Omega_m H_0^2} \Phi(\vec{k}, z) \equiv \mathcal{M}(k, z) \mathcal{R}(k)$

Smoothed density contrast : $\delta_R(z) = \int \frac{d^3k}{(2\pi)^3} \widetilde{W}(kR) \delta(\vec{k}, z)$

Introduction
NG parameters



(a) ε_1, local & equilateral



(b) ε_2, ε_1^2, local

▸ Bispectrum details

Plan

Path integral approach
MR : Maggiore & Riotto, 2009

For now, ignore effects of sharp-x filter, barrier diffusion.
Non-Gaussian halo mass function is ▸ Details

$$f = -2S \frac{\partial}{\partial S}\bigg|_{\delta_c} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_n \exp\left[-\frac{1}{3!} \sum_{j,k,l=1}^{n} \langle \hat{\delta}_j \hat{\delta}_k \hat{\delta}_l \rangle_c \partial_j \partial_k \partial_l + \ldots \right] W^{\rm gm} ,$$

Path integral approach
MR : Maggiore & Riotto, 2009

For now, ignore effects of sharp-x filter, barrier diffusion.
Non-Gaussian halo mass function is ▸ Details

$$f = -2S \frac{\partial}{\partial S}\bigg|_{\delta_c} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_n \exp\left[-\frac{1}{3!} \sum_{j,k,l=1}^{n} \langle \hat{\delta}_j \hat{\delta}_k \hat{\delta}_l \rangle_c \partial_j \partial_k \partial_l + \ldots \right] W^{\rm gm} ,$$

with $W^{\rm gm}$ the "Gaussian-Markovian" p.d.f for the random walk,

$$W^{\rm gm} = \prod_{k=0}^{n-1} \Psi_{\Delta S}(\delta_{k+1} - \delta_k) \ ; \ \Psi_{\Delta S}(x) = \frac{1}{\sqrt{2\pi\Delta S}} e^{-x^2/(2\Delta S)}$$

Path integral approach
MR : Maggiore & Riotto, 2009

For now, ignore effects of sharp-x filter, barrier diffusion.
Non-Gaussian halo mass function is ▸ Details

$$f = -2S\,\frac{\partial}{\partial S}\bigg|_{\delta_c}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_n\,\exp\left[-\frac{1}{3!}\sum_{j,k,l=1}^{n}\langle\,\hat{\delta}_j\hat{\delta}_k\hat{\delta}_l\,\rangle_c\,\partial_j\partial_k\partial_l+\ldots\right]W^{\rm gm}\,,$$

with $W^{\rm gm}$ the "Gaussian-Markovian" p.d.f for the random walk,

$$W^{\rm gm}=\prod_{k=0}^{n-1}\Psi_{\Delta S}(\delta_{k+1}-\delta_k)\ \ ;\ \ \Psi_{\Delta S}(x)=\frac{1}{\sqrt{2\pi\Delta S}}e^{-x^2/(2\Delta S)}$$

MR strategy :

- Linearize in 3-point function $\langle\,\hat{\delta}_j\hat{\delta}_k\hat{\delta}_l\,\rangle_c$
- Taylor expand around $S_n=S$, assuming small S. ▸ Details
- Perform resulting integrals, in continuum limit.

Non-Gaussian mass function is

$$f_{\rm MR}=f_{\rm PS}(\nu)\left[1+\frac{1}{6}\varepsilon_1\nu^3-\frac{1}{4}\varepsilon_1\nu\,(4-c_1)-\frac{\varepsilon_1}{4\nu}\left(c_1-\frac{1}{4}c_2-2\right)\right].$$

Plan

Scale dependence and small parameters

MR's mass function has two small parameters : $\epsilon \sim \langle \hat{\delta}^3 \rangle / \sigma^3$ and $\nu^{-1} \sim \sigma / \delta_c$.

$$f_{\rm MR} \sim f_{\rm PS}(\nu) \left[1 + \epsilon\nu^3 + \epsilon\nu + \epsilon\nu^{-1} + \ldots \right]$$

Scale dependence and small parameters

MR's mass function has two small parameters : $\epsilon \sim \langle \hat{\delta}^3 \rangle / \sigma^3$ and $\nu^{-1} \sim \sigma/\delta_c$.

$$f_{\rm MR} \sim f_{\rm PS}(\nu) \left[1 + \epsilon\nu^3 + \epsilon\nu + \epsilon\nu^{-1} + \ldots \right]$$

Question : Can neglected terms become comparable to those retained?

Scale dependence and small parameters

MR's mass function has two small parameters : $\epsilon \sim \langle \hat{\delta}^3 \rangle / \sigma^3$ and $\nu^{-1} \sim \sigma/\delta_c$.

$$f_{\rm MR} \sim f_{\rm PS}(\nu) \left[1 + \epsilon\nu^3 + \epsilon\nu + \epsilon\nu^{-1} + \ldots \right]$$

Equal-time contribution :

$$\sim S\partial_S \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_n \langle \hat{\delta}(S)^3 \rangle \sum_{j,k,l} \partial_j \partial_k \partial_l W^{\rm gm} \sim \nu\partial_\nu (\varepsilon_1 \partial_\nu^3) {\rm erf}\,(\nu) \sim f_{\rm PS}(\nu)\, \epsilon\nu^3$$

Scale dependence and small parameters

MR's mass function has two small parameters : $\epsilon \sim \langle\,\hat{\delta}^3\,\rangle/\sigma^3$ and $\nu^{-1} \sim \sigma/\delta_c$.

$$f_{\rm MR} \sim f_{\rm PS}(\nu)\left[1 + \epsilon\nu^3 + \epsilon\nu + \epsilon\nu^{-1} + \ldots\right]$$

Equal-time contribution :

$$\sim S\partial_S \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_n \langle\,\hat{\delta}(S)^3\,\rangle \sum_{j,k,l} \partial_j\partial_k\partial_l W^{\rm gm} \sim \nu\partial_\nu(\varepsilon_1\partial_\nu^3)\mathrm{erf}\,(\nu) \sim f_{\rm PS}(\nu)\,\epsilon\nu^3$$

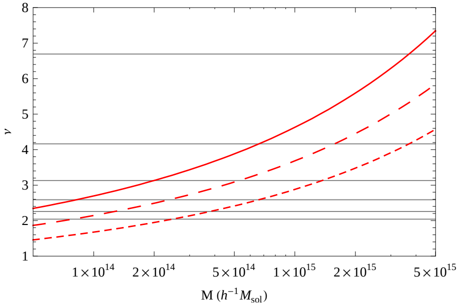
By same logic, expect a term $\sim f_{\rm PS}(\nu)(\epsilon\nu^3)^2$.

This becomes comparable to $\epsilon\nu$ if $\epsilon\nu^3 \sim \nu^{-2}$.

Scale dependence and small parameters

MR's mass function has two small parameters : $\epsilon \sim \langle \hat{\delta}^3 \rangle / \sigma^3$ and $\nu^{-1} \sim \sigma / \delta_c$.

$$f_{\rm MR} \sim f_{\rm PS}(\nu) \left[1 + \epsilon\nu^3 + \epsilon\nu + \epsilon\nu^{-1} + \ldots\right]$$



$\nu \equiv \delta_c(z)/\sigma(M)$
with $\epsilon = 1/300$, for $z = 1$ (solid), $z = 0.5$ (long dashed) and $z = 0$ (short dashed).
Horizontal lines mark ν-values where $\epsilon\nu^3 =$ (from top to bottom) 1, ν^{-1}, ν^{-2}, ν^{-3}, ν^{-4} and ν^{-5}.

An improved mass function
Extending the MR analysis

2 observations :

- $\epsilon\nu^3$ is the worst offender, provided $\epsilon\nu < 1$. (Other terms, e.g. $\epsilon^2\nu^4$ etc., are then *always* parametrically smaller than unity, even for $\epsilon\nu^3 \sim 1$.)

An improved mass function
Extending the MR analysis

2 observations :

- $\epsilon\nu^3$ is the worst offender, provided $\epsilon\nu < 1$. (Other terms, e.g. $\epsilon^2\nu^4$ etc., are then *always* parametrically smaller than unity, even for $\epsilon\nu^3 \sim 1$.)
- $\epsilon\nu^3$ appears as an equal-time term, and can hence be accounted for at all orders.

An improved mass function
Extending the MR analysis

2 observations :

- $\epsilon\nu^3$ is the worst offender, provided $\epsilon\nu < 1$. (Other terms, e.g. $\epsilon^2\nu^4$ etc., are then *always* parametrically smaller than unity, even for $\epsilon\nu^3 \sim 1$.)
- $\epsilon\nu^3$ appears as an equal-time term, and can hence be accounted for at all orders.

$$f = -2S \frac{\partial}{\partial S}\bigg|_{\delta_c} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_n \exp\left[-\frac{1}{3!}\sum_{j,k,l=1}^{n} \langle\, \hat{\delta}_j \hat{\delta}_k \hat{\delta}_l \,\rangle_c \partial_j \partial_k \partial_l + \ldots\right] W^{\mathrm{gm}} ,$$

$$\int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_n \exp\left[-\frac{1}{3!}\langle\, \hat{\delta}(S)^3 \,\rangle \sum_{j,k,l=1}^{n} \partial_j \partial_k \partial_l\right](\ldots) \longrightarrow e^{-(\varepsilon_1/6)\partial_\nu^3} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_n (\ldots)$$

This is true for all equal time terms, e.g. $\sim \langle\, \hat{\delta}(S)^4 \,\rangle_c \sum_{j,k,l,m} \partial_j \partial_k \partial_l \partial_m$, etc.

Unequal-time terms can be handled exactly as in MR calcn.

An improved mass function
Saddle point approximation

Assume ε_1, ε_2 const. for time being. Mass function reduces to the form

$$f \sim \nu\, e^{-(\varepsilon_1/6)\partial_\nu^3 + (\varepsilon_2/24)\partial_\nu^4 + \ldots} \left[e^{-\nu^2/2} (1 + \epsilon\nu + \epsilon\nu^{-1} + \ldots) \right]$$

An improved mass function
Saddle point approximation

Assume ε_1, ε_2 const. for time being. Mass function reduces to the form

$$f \sim \nu\, e^{-(\varepsilon_1/6)\partial_\nu^3 + (\varepsilon_2/24)\partial_\nu^4 + \dots} \left[e^{-\nu^2/2}(1 + \epsilon\nu + \epsilon\nu^{-1} + \dots) \right]$$

$$\longrightarrow \nu \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{2\pi}} e^{i\lambda\nu} e^{-\lambda^2/2 + (-i\lambda)^3\varepsilon_1/6 + (-i\lambda)^4\varepsilon_2/24 + \dots} \mathcal{P}(\lambda)$$

where $\mathcal{P}(\lambda) \sim 1 + i\epsilon\lambda + i\epsilon\lambda^{-1} + \epsilon^2\lambda^2 + \dots$
Unequal-time effects contained in $\mathcal{P}(\lambda)$.

Saddle point calculation can be performed provided $\epsilon\nu < 1$. ▸ Figure

An improved mass function
Finer details

We also account for :

- Barrier diffusion
- Scale dependent errors
- Filter effects

An improved mass function
Finer details

We also account for :

- Barrier diffusion
- Scale dependent errors
- Filter effects

MR's analysis goes through :

- Gaussian case : Reduce 2-d problem to 1-d by change of variable.
- Gaussian case : Final effect is $\delta_c \to \sqrt{q}\delta_c$; $q = (1 + D_B)^{-1} \simeq 0.89^2$ (D_B from sims).
- Non-Gaussian case : Dimensional reasoning implies same effect here as well.

An improved mass function
Finer details

We also account for :

- Barrier diffusion
- Scale dependent errors
- Filter effects

As far as possible, ensure that terms ignored are parametrically *smaller* than terms retained.

In any case, track all possible terms ignored.

This amounts to retaining the structure

$$f(\nu) \sim f_{\rm PS}(\nu)\, e^{\epsilon\nu^3+\epsilon^2\nu^4} \left(1 + \epsilon\nu + \epsilon\nu^{-1} + \mathcal{O}(\epsilon\nu^{-3}, \epsilon^2\nu^2, \epsilon^3\nu^5)\right) \; .$$

An improved mass function
Finer details

We also account for :

- Barrier diffusion
- Scale dependent errors
- Filter effects

Real space top-hat introduces its own non-Markovian/unequal time effects.

$$\langle\,\hat{\delta}(S_j)\hat{\delta}(S_k)\,\rangle = \min(S_j, S_k) + \Delta_{jk}\ .$$

We make same assumptions as MR, reg. filter effects in presence of barrier diffusion.

Gaussian case : $f(\nu) = (1-\tilde{\kappa})f_{\rm PS}(\nu) + \tilde{\kappa}/(2\pi)^{1/2}\nu\Gamma(0,\nu^2/2) + f_{\rm PS}(\nu)\mathcal{O}(\tilde{\kappa}^2)$;
with $\tilde{\kappa} = q(0.464 + 0.002R)$.

Non-Gaussian case : More complicated. "Mixed terms" $\sim\ \tilde{\kappa}\epsilon\nu$ hard to calculate. Issues with "boundary layer", regulation of divergences.

An improved mass function
Final results

Setting $\tilde{\kappa} = 0$

$$f(\nu, t) = f_{\rm PS}(\nu)\ \exp\left[\frac{1}{6}\varepsilon_1\nu^3 - \frac{1}{8}\left(\varepsilon_1^2 - \frac{\varepsilon_2}{3}\right)\nu^4\right] \times \left\{1 - \frac{1}{4}\varepsilon_1\nu\,(4 - c_1) - \frac{\varepsilon_1}{4\nu}\left(c_1 - \frac{1}{4}c_2 - 2\right) + \mathcal{O}(\epsilon^3\nu^5, \epsilon^2\nu^2, \epsilon\nu^{-3})\right\}.$$

c_1, c_2 definitions

An improved mass function
Final results

Setting $\tilde{\kappa} = 0$

$$f(\nu, t) = f_{\rm PS}(\nu) \exp\left[\frac{1}{6}\varepsilon_1\nu^3 - \frac{1}{8}\left(\varepsilon_1^2 - \frac{\varepsilon_2}{3}\right)\nu^4\right] \times \left\{1 - \frac{1}{4}\varepsilon_1\nu\,(4 - c_1) - \frac{\varepsilon_1}{4\nu}\left(c_1 - \frac{1}{4}c_2 - 2\right) + \mathcal{O}(\epsilon^3\nu^5, \epsilon^2\nu^2, \epsilon\nu^{-3})\right\}.$$

▸ c_1, c_2 definitions

Compare MR result

$$f_{\rm MR} = f_{\rm PS}(\nu)\left[1 + \frac{1}{6}\varepsilon_1\nu^3 - \frac{1}{4}\varepsilon_1\nu\,(4 - c_1) - \frac{\varepsilon_1}{4\nu}\left(c_1 - \frac{1}{4}c_2 - 2\right) + \mathcal{O}(\epsilon^2\nu^6, \epsilon\nu^{-3})\right].$$

An improved mass function
Final results

Setting $\tilde{\kappa} = 0$

$$f(\nu,t) = f_{\rm PS}(\nu)\,\exp\left[\frac{1}{6}\varepsilon_1\nu^3 - \frac{1}{8}\left(\varepsilon_1^2 - \frac{\varepsilon_2}{3}\right)\nu^4\right] \times \left\{1 - \frac{1}{4}\varepsilon_1\nu\,(4-c_1) - \frac{\varepsilon_1}{4\nu}\left(c_1 - \frac{1}{4}c_2 - 2\right) + \mathcal{O}(\epsilon^3\nu^5, \epsilon^2\nu^2, \epsilon\nu^{-3})\right\}.$$

c_1, c_2 definitions

Compare Matarrese, Verde & Jiminez (2000) result
$[\dot{v} \equiv d\ln v/d\ln S]$

$$f_{\rm MVJ} = f_{\rm PS}(\nu)\frac{\exp\left[\varepsilon_1\nu^3/6\right]}{(1-\varepsilon_1\nu/3)^{1/2}}\left[1 - \frac{1}{2}\varepsilon_1\nu\left(1 - \frac{2}{3}\dot{\varepsilon}_1\right) + \mathcal{O}(\epsilon\nu)\right].$$

An improved mass function
Final results

Setting $\tilde{\kappa} = 0$

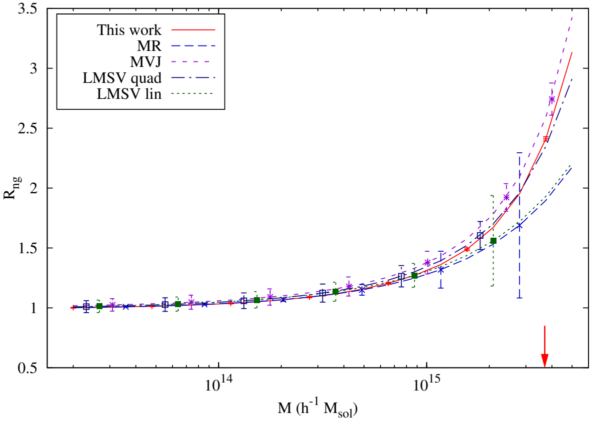
$$f(\nu,t) = f_{\rm PS}(\nu)\ \exp\left[\frac{1}{6}\varepsilon_1\nu^3 - \frac{1}{8}\left(\varepsilon_1^2 - \frac{\varepsilon_2}{3}\right)\nu^4\right] \times \left\{1 - \frac{1}{4}\varepsilon_1\nu\,(4 - c_1) - \frac{\varepsilon_1}{4\nu}\left(c_1 - \frac{1}{4}c_2 - 2\right) + \mathcal{O}(\epsilon^3\nu^5, \epsilon^2\nu^2, \epsilon\nu^{-3})\right\}.$$

c_1, c_2 definitions

Compare Loverde, *et al*. (2008) result
[$\dot{v} \equiv d\ln v / d\ln S$]

$$f_{\rm LMSV,quad} = f_{\rm PS}(\nu)\left[1 + \frac{1}{6}\varepsilon_1\left(H_3(\nu) + \frac{2}{\nu}\dot{\varepsilon}_1 H_2(\nu)\right) + \frac{1}{72}\varepsilon_1^2\left(H_6(\nu) + \frac{4}{\nu}\dot{\varepsilon}_1 H_5(\nu)\right) + \frac{1}{24}\varepsilon_2\left(H_4(\nu) + \frac{2}{\nu}\dot{\varepsilon}_2 H_3(\nu)\right) + \mathcal{O}(\epsilon\nu, \epsilon^3\nu^9)\right].$$

An improved mass function
Final results

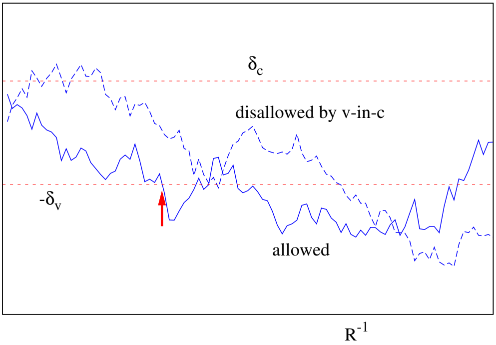


$f(\nu, S, f_{\rm NL} = 100)/f(\nu, S, f_{\rm NL} = 0)$; for $z = 1$ and $\tilde{\kappa} = 0$.

Plan

Voids and excursion sets
Sheth & van de Weygaert, 2004

- In spherical collapse model, a good definition of “void formation” is “first shell crossing”.
- Excursion sets + spherical collapse $\implies$ this occurs when $\hat{\delta}_R < -\delta_v$; where $\delta_v \simeq 2.7$.
- But “voids in clouds” are not real voids, since they would be crushed by the collapsing cloud.
- So first crossing of $(-\delta_v)$ must occur *before* first crossing of $(+\delta_c)$.



Voids and excursion sets
Sheth & van de Weygaert, 2004

- In spherical collapse model, a good definition of “void formation” is “first shell crossing”.
- Excursion sets + spherical collapse $\implies$ this occurs when $\hat{\delta}_R < -\delta_v$; where $\delta_v \simeq 2.7$.
- But “voids in clouds” are not real voids, since they would be crushed by the collapsing cloud.
- So first crossing of $(-\delta_v)$ must occur *before* first crossing of $(+\delta_{\rm c})$.
- Leads to

$$\mathcal{F}_{\rm SvdW}(S) = \sum_{j=1}^{\infty} \frac{j\pi}{\delta_T^2} \sin\left(\frac{j\pi\delta_{\rm v}}{\delta_{\rm T}}\right) \exp\left(-\frac{j^2\pi^2 S}{2\delta_{\rm T}^2}\right) \quad ; \quad \delta_{\rm T} \equiv \delta_{\rm v} + \delta_{\rm c} \,.$$

Voids and excursion sets
Sheth & van de Weygaert, 2004

- In spherical collapse model, a good definition of "void formation" is "first shell crossing".
- Excursion sets + spherical collapse $\implies$ this occurs when $\hat{\delta}_R < -\delta_v$; where $\delta_v \simeq 2.7$.
- But "voids in clouds" are not real voids, since they would be crushed by the collapsing cloud.
- So first crossing of $(-\delta_v)$ must occur *before* first crossing of $(+\delta_c)$.
- Leads to

$$\mathcal{F}_{\rm SvdW}(S) = \sum_{j=1}^{\infty} \frac{j\pi}{\delta_T^2} \sin\left(\frac{j\pi\delta_{\rm v}}{\delta_{\rm T}}\right) \exp\left(-\frac{j^2\pi^2 S}{2\delta_{\rm T}^2}\right) \quad ; \quad \delta_{\rm T} \equiv \delta_{\rm v} + \delta_{\rm c} \,.$$

- Convenient to re-organize the series as

$$\mathcal{F}_{\rm SvdW}(S) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \frac{\Delta_n}{S^{3/2}} \exp\left(-\frac{\Delta_n^2}{2S}\right) \quad ; \quad \Delta_n \equiv \delta_{\rm v} - 2n\delta_{\rm T} \,.$$

Voids and excursion sets : non-Gaussianities
Void-in-cloud corrections

- In Gaussian case, with small S

$$f_{\rm SvdW} = 2S\mathcal{F}_{SvdW} = \sum_{n=-\infty}^{\infty} f_{\rm PS}\left(\frac{\Delta_n}{\sqrt{S}}\right) \longrightarrow f_{\rm PS}(\delta_v/\sqrt{S}) + \ldots$$

Pictorially, we are in the extreme tails of Gaussian p.d.f.'s with means $2n\delta_T$ – hence the nearest Gaussian ($n = 0$) gives biggest contrib$^{\rm n}$.

- One could now argue : since PS works for Gaussian small S, it should also work for *non*-Gaussian small S [Kamionkowski, Verde & Jimenez, 2008]. Essentially just replace $\delta_c \to -\delta_v$.

Voids and excursion sets : non-Gaussianities
Void-in-cloud corrections

- However, care is needed since extreme tails are now strongly non-Gaussian. Path integral formalism allows a more careful calculation. Final result for two fixed barriers is

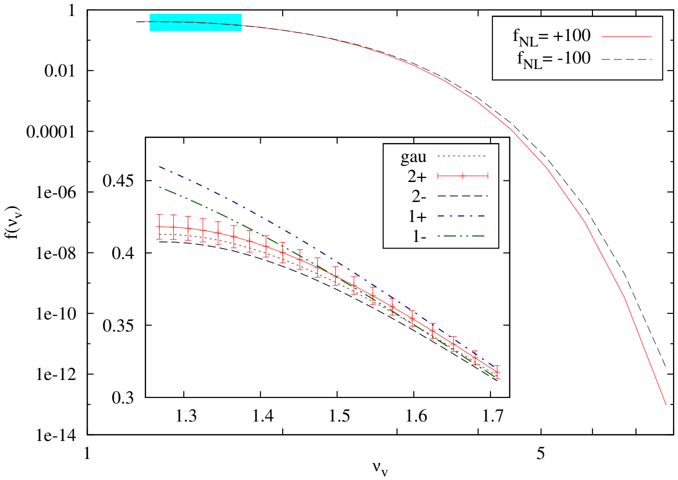
$$f_{\rm 2barrier,NG}(\nu_{\rm v},\nu_{\rm T}) = \sum_{n=-\infty}^{\infty} f_{\rm 1barrier,NG}\left(\nu_{\rm v} - 2n\nu_{\rm T}\right) \;\; ; \;\; \nu = \delta/\sqrt{S}\,.$$

$$f_{\rm 1barrier,NG}(\nu) = f_{\rm PS}(\nu)\exp\left[-\frac{1}{6}\varepsilon_1\nu^3 - \frac{\nu^4}{8}\left(\varepsilon_1^2 - \frac{\varepsilon_2}{3}\right) + \ldots\right]\left(1 - \frac{1}{4}\varepsilon_1\nu(c_1 - 4) + \ldots\right)\,.$$

- Effective $|\epsilon\nu|$ becomes > 1 already at $n = \pm 1$ for $z = 0$, $M \sim 10^{14}h^{-1}M_{\rm sol}$. Series expansion therefore breaks down. However, this can be argued to occur **after** Gaussian suppression due to $f_{\rm PS}$ has kicked in.
- Similar reasoning shows v-in-c is relevant only for small masses.
- In fact, to very good accuracy,

$$f_{\rm 2barrier,NG}(\nu_{\rm v},\nu_{\rm T}) = f_{\rm SvdW}(\nu_{\rm v},\nu_{\rm T}) \times \frac{f_{\rm 1barrier,NG}(\nu_{\rm v})}{f_{\rm PS}(\nu_{\rm v})}$$

Voids and excursion sets : non-Gaussianities
Final results



Summary

- Halo mass function
  - Path integral formalism can be combined with saddle point techniques.
  - Resulting mass function expected to be valid at high redshifts for large masses.
  - Scale dependent theoretical errors can be tracked (enabling comparison between different calculations).
- Void mass function
  - Void-in-cloud issue makes the problem more challenging.
  - Non-Gaussian m.f. can be formally written as sum of infinite n$^{o.}$ of single barrier m.f.'s.
  - In practice, non-Gaussian m.f. $=$ single barrier ratio $\times$ SvdW m.f.
  - V-in-c effects negligible for large voids, $\sim \mathcal{O}(10\%)$ for smaller voids.
  - Still some ways to go before this can be applied to observations.

Summary

- Halo mass function
  - Path integral formalism can be combined with saddle point techniques.
  - Resulting mass function expected to be valid at high redshifts for large masses.
  - Scale dependent theoretical errors can be tracked (enabling comparison between different calculations).
- Void mass function
  - Void-in-cloud issue makes the problem more challenging.
  - Non-Gaussian m.f. can be formally written as sum of infinite n$^{\text{o.}}$ of single barrier m.f.'s.
  - In practice, non-Gaussian m.f. $=$ single barrier ratio $\times$ SvdW m.f.
  - V-in-c effects negligible for large voids, $\sim \mathcal{O}(10\%)$ for smaller voids.
  - Still some ways to go before this can be applied to observations.

Thank you.

Connected three point function of primordial curvature perturbation is characterised by the "bispectrum" $B_{\mathcal{R}}(k_1, k_2, k_3)$:

$$\langle\, \mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\,\rangle_c = (2\pi)^3 \delta_D(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B_{\mathcal{R}}(k_1, k_2, k_3)\,.$$

Local NG bispectrum is peaked on squeezed triangles $k_1 \ll k_2 \simeq k_3$:

$$B_{\mathcal{R}}(k_1, k_2, k_3) = \frac{6}{5} f_{\rm NL}^{\rm loc} \left[P_{\mathcal{R}}(k_1) P_{\mathcal{R}}(k_2) + {\rm cycl.}\right] \quad ; \quad P_{\mathcal{R}}(k) = A k^{n_s - 4}\,.$$

Equilateral NG bispectrum is peaked on equilateral triangles $k_1 \simeq k_2 \simeq k_3$:

$$B_{\mathcal{R}}(k_1, k_2, k_3) = \frac{18}{5} f_{\rm NL}^{\rm equil} A^2 \left[\frac{1}{2k_1^{4-n_s} k_2^{4-n_s}} + \frac{1}{3(k_1 k_2 k_3)^{2(4-n_s)/3}} - \frac{1}{(k_1 k_2^2 k_3^3)^{(4-n_s)/3}} + 5 \text{ perms.}\right]\,.$$

Back

Path integral approach
MR, 2009

Excursion set formalism derived from first principles.
The random variable $\hat{\delta}(S)$ obeys a Langevin equation $\partial\hat{\delta}/\partial S = \hat{\eta}$.
For sharp k-space filter, noise is white (process is Markovian) : $\langle\,\hat{\eta}(S_1)\hat{\eta}(S_2)\,\rangle = \delta_{\rm D}(S_1 - S_2)\,.$

Start with p.d.f. for random walk, with n discrete steps ΔS from $S_0 = 0$ to $S_n = n\Delta S \equiv S$:

$$W(\{\delta_j\};S) \equiv \langle\,\delta_{\rm D}(\hat{\delta}(S_1)-\delta_1)\ldots\delta_{\rm D}(\hat{\delta}(S_n)-\delta_n)\,\rangle$$
$$= \int_{-\infty}^{\infty}\frac{d\lambda_1}{2\pi}\ldots\frac{d\lambda_n}{2\pi}\langle\,e^{-i\sum_j\lambda_j\hat{\delta}(S_j)}\,\rangle e^{i\sum_j\lambda_j\delta_j}\,,$$

in which we use $\langle\,e^{-i\sum_j\lambda_j\hat{\delta}(S_j)}\,\rangle = \exp\left[\sum_{p=2}^{\infty}\frac{(-i)^p}{p!}\sum_{j_1,\ldots,j_p=1}^{n}\lambda_{j_1}\ldots\lambda_{j_p}\langle\,\hat{\delta}(S_{j_1})\ldots\hat{\delta}(S_{j_p})\,\rangle_c\right]$.

$$P(\hat{S}_f > S) \equiv \text{Probab. that first crossing time } \hat{S}_f > S$$
$$= \text{Probab. that barrier not crossed until time } S$$
$$= \lim_{\Delta S\to 0}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_n W(\{\delta_j\};S)$$

First crossing distribution : $\mathcal{F}(S) = -\partial_S P(\hat{S}_f > S)$.
Mass function : $f(S) = 2S\mathcal{F}(S)$.

Back

Gaussian case :

$\langle e^{-i\sum_j \lambda_j \hat{\delta}(S_j)} \rangle = \exp\left[-\frac{1}{2}\sum_{j_1,j_2=1}^{n} \lambda_{j_1}\lambda_{j_2}\langle \hat{\delta}_{j_1}\hat{\delta}_{j_2} \rangle\right]$.

For sharp-k filter : $\langle \hat{\delta}_j \hat{\delta}_k \rangle = \min(S_j, S_k)$, and

$$W(\{\delta_j\};S) = W^{\rm gm} = \prod_{k=0}^{n-1} \Psi_{\Delta S}(\delta_{k+1} - \delta_k) \;\; ; \;\; \Psi_{\Delta S}(x) = \frac{1}{\sqrt{2\pi\Delta S}} e^{-x^2/(2\Delta S)}$$

In continuum limit, MR show (non-trivially) that this recovers $f_{\rm PS}(\nu)$.

Non-Gaussian case :

Use $\lambda_k e^{i\sum_j \lambda_j \delta_j} = -i\partial_k e^{i\sum_j \lambda_j \delta_j}$, leading to

$$f = -2S \left.\frac{\partial}{\partial S}\right|_{\delta_c} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_n \exp\left[-\frac{1}{3!}\sum_{j,k,l=1}^{n} \langle \hat{\delta}_j \hat{\delta}_k \hat{\delta}_l \rangle_c \partial_j \partial_k \partial_l + \ldots\right] W^{\rm gm} ,$$

Back

Taylor expansion for 3-point function

$$\langle\, \hat\delta_j \hat\delta_k \hat\delta_l \,\rangle_c = \sum_{p,q,r=0}^{\infty} \frac{(-1)^{p+q+r}}{p!q!r!} \mathcal{G}_3^{(p,q,r)}(S)(S-S_j)^p(S-S_k)^q(S-S_l)^r$$

$$\mathcal{G}_3^{(p,q,r)}(S) \equiv \left[\frac{d^p}{dS_j^p}\frac{d^q}{dS_k^q}\frac{d^r}{dS_l^r} \langle\, \hat\delta(S_j)\hat\delta(S_k)\hat\delta(S_l) \,\rangle_c \right]_{S_j=S_k=S_l=S}$$

$$\mathcal{G}_3^{(1,0,0)} = \frac{1}{2}\varepsilon_1(S)c_1(S)S^{1/2} \;\; ; \;\; \mathcal{G}_3^{(2,0,0)} = -\frac{1}{4}\varepsilon_1(S)c_2(S)S^{-1/2} \,,$$

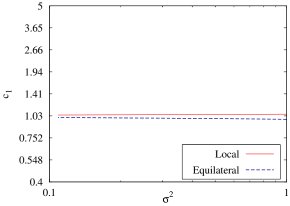
$$\mathcal{G}_3^{(1,1,0)} = \frac{1}{4}\varepsilon_1(S)c_3(S)S^{-1/2}$$

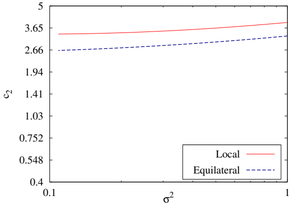
Also, with $\dot\varepsilon_1 \equiv d\ln\varepsilon_1/d\ln S$, etc.

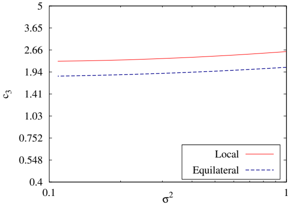
$$\dot\varepsilon_1 = \frac{3}{2}(c_1-1) \;\; ; \;\; \dot c_1 = 1 - \frac{3}{2}c_1 + \frac{1}{c_1}\left(c_3 - \frac{1}{2}c_2\right) \,.$$

◂ MR calcn
◂ Final results

The c_n



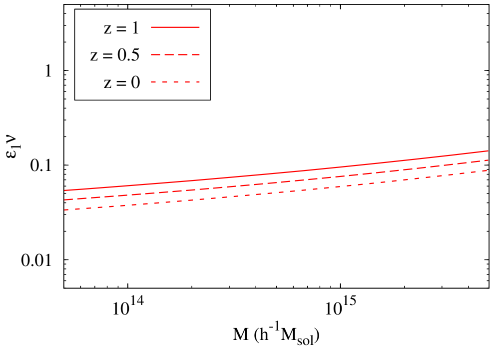




◂ MR calcn

◂ Final results

Behaviour of $\varepsilon_1 \nu$
Local model, $f_{\rm NL} = 100$



Back