On the squeezed limit of the bispectrum in general single field inflation

Sébastien Renaux-Petel

CTC, DAMTP, Cambridge

PFNG

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Outline

I : Motivations

II : The theorem and its implications

III : The original proof

IV : … and its weaknesses

V : Recent results

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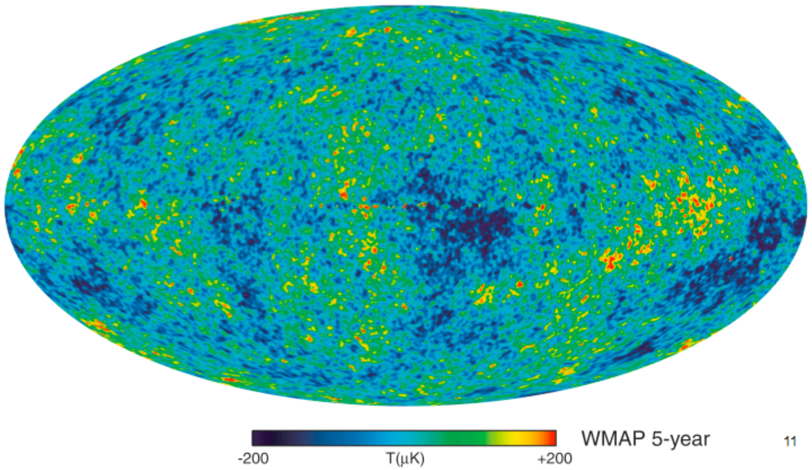
Cosmological inflation

• A period of accelerated expansion before the radiation era that solves the problems of the Hot Big-Bang model and creates the seeds of the large scale structure of the universe.



• Primordial fluctuations adiabatic

$$\frac{\delta T}{T} = -\frac{\zeta}{5}$$

← Primordial curvature perturbation

• nearly scale invariant
• Gaussian

• Simplest implementation: single field with very flat potential. Its predictions perfectly match the observations.

More?

- Simplest models surprisingly difficult to embed in high-energy physics models (eta-problem).

- Many high energy physics models involve several scalar fields. If several scalar fields are light enough during inflation
  ⇨ **multifield inflation**, changes a lot the predictions !

- D-brane action: **non-standard kinetic terms**.

- Alternatives: curvaton, ekpyrotic…

- They are all almost degenerate at the linear level.

(cook up a model that matches two numbers…)

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How to discriminate
amongst them?

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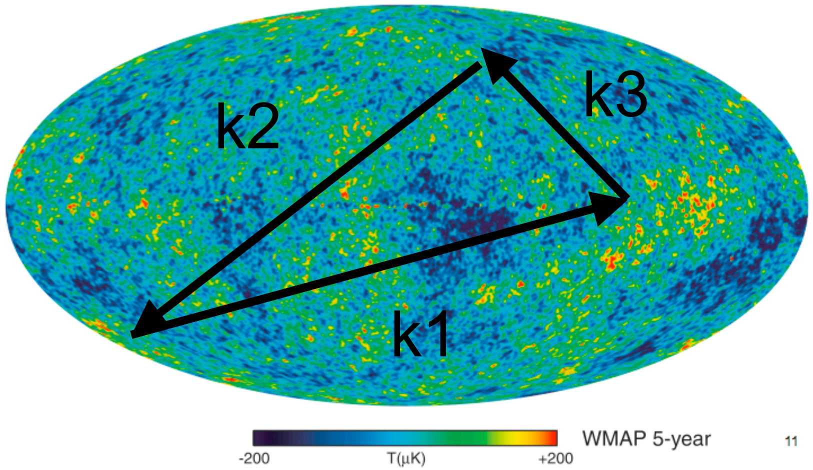
How to discriminate amongst them?

NON GAUSSIANITIES

Non-Gaussianities

Beyond the power spectrum:
higher-order, connected,
n-point functions.



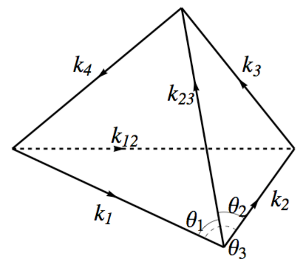
3-point function, the bispectrum

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle \equiv B_\zeta(k_1, k_2, k_3)(2\pi)^3 \delta^3(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3})$$

$$\equiv \frac{6}{5} f_{NL} \left[P_\zeta(k_1) P_\zeta(k_2) + \text{perm.} \right]$$

Connected 4-point function of zeta, the trispectrum

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \zeta_{\mathbf{k}_4} \rangle_c \equiv T_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)(2\pi)^3 \delta^3 \left(\sum_{\mathbf{i}} \mathbf{k_i} \right)$$



The Bispectrum

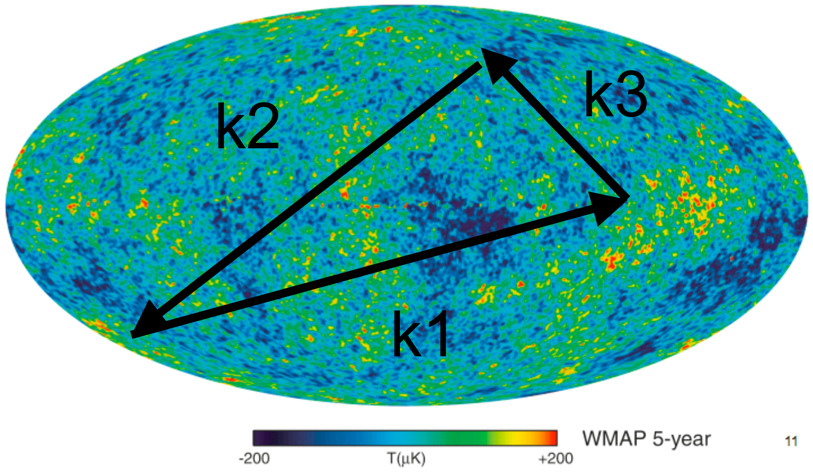
$$B_\zeta(k_1, k_2, k_3)$$

- Amplitude

A useful guidance: local-type non-Gaussianities

$$\zeta = \zeta_G\left(1 + f_{NL}\zeta_G\right)$$

Current constraints $f_{NL} = O(100)$
Planck accuracy $\Delta f_{NL} \sim 5$
Slow-roll single field $f_{NL} \sim 10^{-2}$



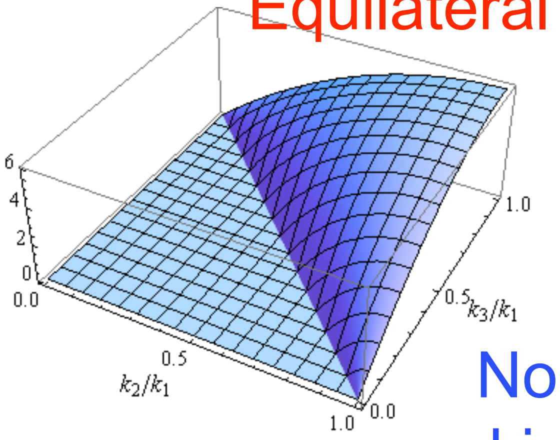
Each feature can rule out large classes of models

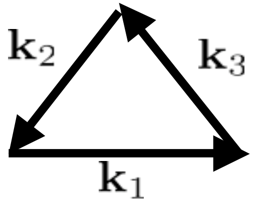
- Scale-dependence (growing or shrinking on small scales?)
- Sign (more or less cold spots?)
- Shape (largest for which triangles?)

Babich et al (04)
Fergusson & Shellard (08)

Various shapes

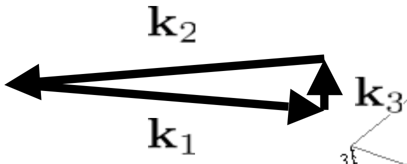
Equilateral

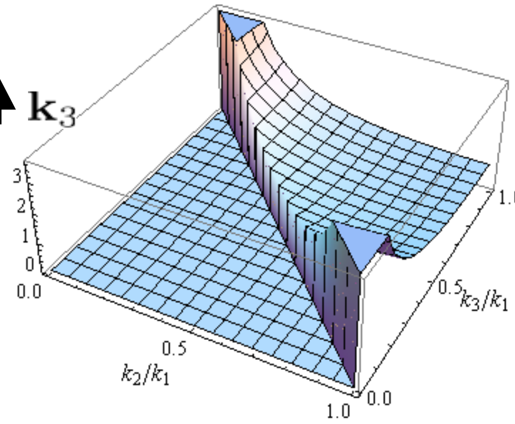




Non standard kinetic terms

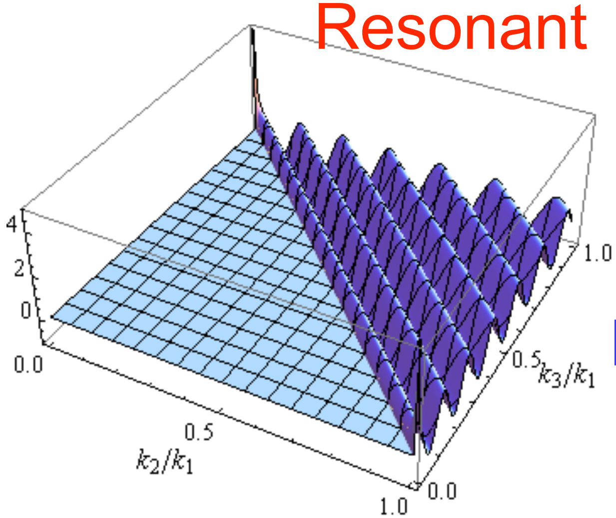
Local





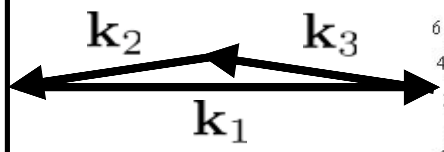
Multiple fields

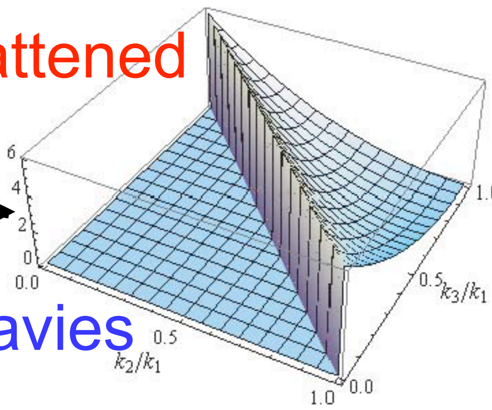
Resonant



Periodic background evolution

Flattened





(a)

Non Bunch-Davies vacuum

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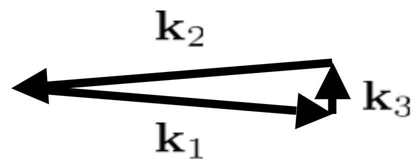
The single field consistency relation

- In every single field model, irrespective of kinetic terms, potential, vacuum, slow roll …:

Maldacena (03), Creminelli & Zaldarriaga (04)

$$f_{NL}^{sq}(k_1) = \frac{5}{12}(1 - n_s(k_1))$$

with $$f_{NL}^{sq}(k_1) \equiv \lim_{k_3 \to 0} f_{NL}(k_1, k_2, k_3)$$



- Given that $$n_s = 0.963 \pm 0.012\,(68\% CL)$$

WMAP 7th year (2010)

- If $f_{NL}^{sq} \gtrsim 1$ is robustly detected, all single field models would be ruled out!

- Current constraints: $$f_{NL}^{loc} = 32 \pm 21\,(68\%\,CL)$$

Understanding the theorem (1)

- In the squeezed limit, one correlates one very long wavelength mode with two shorter wavelength modes

$$k_l = k_3 \qquad\qquad k_s = k_1 \approx k_2$$

$$\Rightarrow \quad \langle \zeta_{\boldsymbol{k}_1} \zeta_{\boldsymbol{k}_2} \zeta_{\boldsymbol{k}_3} \rangle \approx \langle (\zeta_{\boldsymbol{k}_s})^2 \zeta_{\boldsymbol{k}_l} \rangle$$

- Question: why $(\zeta_{\boldsymbol{k}_s})^2$ should care about $\zeta_{\boldsymbol{k}_l}$?

- The theorem says it does not care if $\zeta_{\boldsymbol{k}}$ is exactly scale-invariant.

Understanding the theorem (2)

- A very long wavelength mode acts as a local rescaling of the spatial coordinates (equivalently, of the scale factor)

$$ds^2 \simeq -dt^2 + a(t)^2 e^{2\zeta_l} (d\boldsymbol{x})^2$$

$\Rightarrow$

$$\boldsymbol{x} \to \boldsymbol{x}\, e^{\zeta_l}$$

- A conformal rescaling of the spatial coordinates can not change the amplitude of the small-scale perturbation if $\zeta_{\boldsymbol{k}}$ is scale invariant.

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A formal proof

Creminelli & Zaldarriaga (04)
Cheung et al (08)

1) Local rescaling of coordinates

$$\langle \zeta\zeta \rangle_l (\mathbf{x}_1, \mathbf{x}_2) \simeq \langle \zeta\zeta \rangle_0 (e^{\zeta_l} |\mathbf{x}_1 - \mathbf{x}_2|)$$

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$$\langle\zeta\zeta\rangle_l(\mathbf{x}_1,\mathbf{x}_2) \simeq \langle\zeta\zeta\rangle_0(e^{\zeta_l}|\mathbf{x}_1-\mathbf{x}_2|)$$

2) Linear expansion in ζ_l

$$\langle\zeta\zeta\rangle_l(\mathbf{x}_1,\mathbf{x}_2) \simeq \langle\zeta\zeta\rangle_0(|\mathbf{x}_1-\mathbf{x}_2|) + \zeta_l\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}\right)\frac{d}{d\log(|\mathbf{x}_1-\mathbf{x}_2|)}\langle\zeta\zeta\rangle_0(|\mathbf{x}_1-\mathbf{x}_2|)$$

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3) Algebra

$$\langle\zeta\zeta\rangle_l(\mathbf{k}_1,\mathbf{k}_2) \simeq \langle\zeta\zeta\rangle_0(\mathbf{k}_s) - \zeta_l(\mathbf{k}_l)\frac{1}{k_s^3}\frac{d}{d\log k_s}\left[k_s^3\langle\zeta\zeta\rangle_0(k_s)\right]$$

with $\quad \mathbf{k}_l \equiv \mathbf{k}_1+\mathbf{k}_2 \qquad \mathbf{k}_s \equiv \frac{\mathbf{k}_1-\mathbf{k}_2}{2} \simeq \mathbf{k}_1 \simeq -\mathbf{k}_2$

$$\langle\zeta_l(\mathbf{k}_3)\langle\zeta\zeta\rangle_l(\mathbf{k}_1,\mathbf{k}_2)\rangle \simeq -(2\pi)^3\delta^{(3)}(\sum \mathbf{k}_i)P_\zeta(k_3)P_\zeta(k_1)\frac{1}{k_1^3}\frac{d}{d\log k_1}\left[k_1^3\langle\zeta\zeta\rangle_0(k_1)\right]$$

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Some concern

Chen (10)

- Loop corrections

- Finite k_3/k_1 corrections:

real triangles are not infinitely squeezed

- The proof uses classical arguments only whereas correlation functions are calculated in a quantum set-up:

we assumed that the effect of of the long-wavelength mode is a constant background rescaling, i.e. we assume there is no interaction when all modes are within the horizon.

Cosmological correlations from quantum field theory and primordial non-Gaussianities

- Keldysh-Schwinger formalism

Schwinger (61), Keldysh (64), Weinberg (05)

$$\langle O(t)\rangle = \langle 0| \left[\bar{T}\exp\left(i\int_{-\infty}^{t} H_I(t')dt'\right)\right] O^I(t) \left[T\exp\left(-i\int_{-\infty}^{t} H_I(t'')dt''\right)\right]|0\rangle \ .$$

I = interaction picture.

Interaction Hamiltonian

- All fields are free Gaussian fields

- Requires integrals over time:

The oscillations of the mode functions inside the horizon are usually destructive, but not always: models with a step, periodic background evolution … Chen et al (06,08)

- One expects the theorem to be satisfied for

$$k_3 \ll k_* < k_1 \approx k_2$$ where k_* is any characteristic scale.

Example: Flauger & Pajer (10)

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Recent results (1)

- In a quantum framework, exact (tree level) expression of the squeezed bispectrum (in terms of integrals of the free mode functions) for standard single field inflation. Ganc & Komatsu (2010)

- Generalization to k-inflation: Renaux-Petel (2010)

$P(X, \phi)$ with $X = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi$

  - Standard kinetic term: $P = X - V(\phi)$

  - Non trivial example: DBI $P = -\frac{1}{f(\phi)}\left(\sqrt{1 - 2f(\phi)X} - 1\right) - V(\phi)$

Recent results (2)

Explicit verification of the consistency relation:

• for an exactly solvable class of models with a non-trivial speed of sound.

• at first non trivial order in a slow-varying approximation in k-inflation (a known result)

Chen et al (2006)
Cheung et al (2007)

• at second order in a slow-varying approximation in standard single field inflation.

The strategy (1)

- Calculate the squeezed limit of the bispectrum without first calculating the full bispectrum

- We still calculate $\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = \langle \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle_{\mathbf{k}_3} \zeta_{\mathbf{k}_3} \rangle$

- For that purpose, we split zeta into a large-scale, classical, part and a small-scale quantum part:

$$\zeta_l \equiv \int_{k<k_*} \frac{dk^3}{(2\pi)^3} \zeta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \zeta_s \equiv \int_{k>k_*} \frac{dk^3}{(2\pi)^3} \zeta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

with $k_3 < k_* \ll k_1 \simeq k_2$

The strategy (2)

- We introduce $\zeta = \zeta_l + \zeta_s$ into the second- and third-order action

- Terms of order ζ_s^2 (from the second-order action)

⇨ free mode functions

- Terms of order $\zeta_s^2 \zeta_l$ (from the third-order action) are treated as perturbations in the Keldysh-Schwinger formalism

- In the infinitely squeezed limit $k_3 \to 0$ one can neglect time and space derivatives of the long wavelength mode.

Action

- With the redefined field we obtain $$\zeta_s = \zeta_n + \frac{\eta}{2c_s^2}\zeta_n + \dots$$

$$S_0 = \int dt d^3x \, [a^3 \frac{\epsilon}{c_s^2} \dot{\zeta_n}^2 - a\epsilon(\partial\zeta_n)^2] \, ,$$

$$S_{\text{int},(3)} = \int dt d^3x \left[\frac{a^3\epsilon}{c_s^4}(\epsilon - 3 + 3c_s^2)\zeta_l \dot{\zeta_n}^2 + \frac{a\epsilon}{c_s^2}(\epsilon - 2s + 1 - c_s^2)\zeta_l(\partial\zeta_n)^2 + \frac{a^3\epsilon}{c_s^2}\left(\frac{\eta}{c_s^2}\right)^{\dot{}} \zeta_l \zeta_n \dot{\zeta_n} \right]$$

with $$c_s^2 \equiv \frac{P_{,X}}{P_{,X} + 2XP_{,XX}}$$

and $$\epsilon \equiv -\frac{\dot{H}}{H^2} \, , \qquad \eta \equiv \frac{\dot{\epsilon}}{H\epsilon} \, , \qquad s \equiv \frac{\dot{c}_s}{Hc_s}$$

Quantization

Mode expansion

$$\zeta_n(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left(u_k(t)\hat{a}_{\mathbf{k}} + u_k^*(t)\hat{a}^\dagger_{-\mathbf{k}} \right)$$

Canonically normalized field

$$v_k \equiv \frac{a\sqrt{2\epsilon}}{c_s} u_k = z u_k$$

$$v_k'' + \left(c_s^2 k^2 - \frac{z''}{z} \right) v_k = 0$$

$$v_k^* v_k' - v_k v_k'^* = -i$$

Power spectrum

$$P_\zeta(k) = \lim_{aH \gg k c_s} |u_k|^2$$

Interactions and general result

Tree level Keldysh-Schwinger formalism:

$$\langle \zeta_{n,\mathbf{k}_1}(\bar{t})\zeta_{n,\mathbf{k}_2}(\bar{t})\rangle_{\zeta_{l,\mathbf{k}_3}} = -i\int_{-\infty}^{t} dt \langle 0|\zeta_{n,\mathbf{k}_1}(\bar{t})\zeta_{n,\mathbf{k}_2}(\bar{t})H_{I,(3)}(t)|0\rangle + \text{c.c.}$$

$$\Rightarrow \quad \langle \zeta_{n,\mathbf{k}_1}(\bar{t})\zeta_{n,\mathbf{k}_2}(\bar{t})\rangle_{\zeta_{l,\mathbf{k}_3}} = F\zeta_{l,\mathbf{k}_1+\mathbf{k}_2}$$

$$\begin{aligned} F \quad = \quad & iu_{k_1}^2(\bar{\tau})\int_{-\infty}^{\bar{\tau}} d\tau \left[\frac{2\epsilon}{c_s^4}(\epsilon - 3 + 3c_s^2)a^2(u'^*_{k_1})^2 + \frac{2\epsilon}{c_s^2}(1 - c_s^2 + \epsilon - 2s)a^2k_1^2(u^*_{k_1})^2 \right. \\ & \left. + \frac{2\epsilon}{c_s^2}\left(\frac{\eta}{c_s^2}\right)^{\dot{}} a^3 u'^*_{k_1} u^*_{k_1}\right] + \text{c.c.} \end{aligned}$$

Final result:

$$\lim_{k_3 \to 0}\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = (2\pi)^3\delta^3(\sum_i \mathbf{k}_i)P_\zeta(k_3)(P_\zeta(k_1)\frac{\eta(\bar{t})}{c_s^2(\bar{t})} + F)$$

The slow varying approximation (1)

$$\epsilon, \eta, s = \mathcal{O}(\epsilon) \ll 1$$ and $$\frac{1}{H}(\dot{\epsilon}, \dot{\eta}, \dot{s}) = \mathcal{O}(\epsilon^2)$$

$$u_k(y) = i\frac{\sqrt{\pi}}{2\sqrt{2}}\,\frac{H}{\sqrt{\epsilon c_s}}\,\frac{1}{k^{3/2}}(1+\frac{\epsilon}{2}+\frac{s}{2})\;y^{3/2}H_\nu^{(1)}\left((1+\epsilon+s)y\right)\left(1+\mathcal{O}(\epsilon^2)\right)$$

Chen et al (06)

with $$y \equiv \frac{kc_s}{aH}$$ and $$\nu \equiv \frac{3}{2}+\epsilon+\frac{\eta}{2}+\frac{s}{2}$$

⇨ Scalar spectral index at second order

$$u_k(\tau) = \frac{H_K}{\sqrt{4\epsilon_K c_{sK} k^3}}(1+ikc_{sK}\tau)e^{-ikc_{sK}\tau} \qquad 0^{th}\text{ order}$$

The slow varying approximation (2)

$$
\begin{aligned}
F &= F_1 + F_2 + F_3 \qquad \text{with} \\
F_1 &= 4P_\zeta(k_1)\,\mathrm{Re}\left[-i\int_{-\infty}^{\bar{\tau}} d\tau\, g_1(\tau) a^2 (u'^*_{k_1})^2\right],\ g_1(\tau) = \frac{\epsilon}{c_s^4}(3 - 3c_s^2 - \epsilon) \\
F_2 &= 4P_\zeta(k_1)\,\mathrm{Re}\left[-i\int_{-\infty}^{\bar{\tau}} d\tau\, g_2(\tau) a^2 k_1^2 (u^*_{k_1})^2\right],\ g_2(\tau) = -\frac{\epsilon}{c_s^2}(1 - c_s^2 + \epsilon - 2s) \\
F_3 &= 4P_\zeta(k_1)\,\mathrm{Re}\left[-i\int_{-\infty}^{\bar{\tau}} d\tau\, g_3(\tau) a^3 u'^*_{k_1} u^*_{k_1}\right],\ g_3(\tau) = -\frac{\epsilon}{c_s^2}\left(\frac{\eta}{c_s^2}\right)^{\cdot}
\end{aligned}
$$

g_3 is higher order in the slow varying approximation

A warm-up: canonical inflation at leading order

- Coupling treated as constants
- Zeroth-order mode functions

$$\frac{F_1}{P_\zeta(k)} = -\epsilon_K\, k \mathrm{Re}\left[-i \int_{-\infty}^{\tau} d\tau e^{2ik\tau}\right]$$

$$\frac{F_2}{P_\zeta(k)} = -\frac{\epsilon_K}{k} \mathrm{Re}\left[-i \int_{-\infty}^{\bar{\tau}} \frac{d\tau}{\tau^2}(1-ik\tau)^2 e^{2ik\tau}\right]$$

$$\begin{aligned}\lim_{k_3\to 0}\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle &= -(2\pi)^3\delta^{(3)}(\sum_i \mathbf{k}_i)P_\zeta(k_1)P_\zeta(k_3)\left(-2\epsilon_K - \eta_K + \mathcal{O}(\epsilon^2)\right) \\ &= -(2\pi)^3\delta^{(3)}(\sum_i \mathbf{k}_i)P_\zeta(k_1)P_\zeta(k_3)\left(-2\epsilon_{k_1} + \eta_{k_1} - \mathcal{O}(\epsilon^2)\right)\end{aligned}$$

⇨ The theorem is verified

The case of an arbitrary speed of sound at leading order (1)

- Same type of calculation:

$$\frac{F_{\rm naive}}{P_\zeta(k)} = \frac{2\epsilon - 3s}{c_s^2}$$

- But the calculation is not consistent at this stage: by treating all the slow-varying parameters as constant, we neglected $\mathcal{O}(\epsilon)$ corrections, which, multiplied by the $1/c_s^2 - 1$ factor in g1 and g2, compete with the above result.

The case of an arbitrary speed of sound at leading order (2)

• 3 types of $\mathcal{O}(\epsilon)$ corrections:

- to the scale factor:

$$a(\tau) = -\frac{1}{H_K \tau} - \frac{\epsilon}{H_K \tau} + \frac{\epsilon}{H_K \tau} \ln(\tau/\tau_K) + \mathcal{O}(\epsilon^2)$$

- to the coupling « constants »:

$$g(\tau) = g(\tau_K) - \frac{dg}{dt} \frac{1}{H_K} \ln \frac{\tau}{\tau_K} + \mathcal{O}(\epsilon^2 g)$$

- to the mode functions themselves:
complicated!

The case of an arbitrary speed of sound at leading order (3)

Lots of integrals … with special functions derived from Hankel functions …

$$
\begin{aligned}
\frac{\eta(\bar{\tau})}{c_s^2(\bar{\tau})} + \frac{F}{P_\zeta(k)} &= \left(\frac{\eta_K}{c_{sK}^2} + \frac{F_{\text{naive}}}{P_\zeta(k)} + \frac{\Delta F}{P_\zeta(k)}\right)(1+\mathcal{O}(\epsilon)) \\
&= \left(\frac{\eta_K}{c_{sK}^2} + \frac{2\epsilon_K - 3s_K}{c_{sK}^2} - \left(\frac{1}{c_{sK}^2} - 1\right)(2\epsilon_K + \eta_K - 3s_K) + 4s_K\right)(1+\mathcal{O}(\epsilon)) \\
&= 2\epsilon_K + \eta_K + s_K + \mathcal{O}(\epsilon^2) \\
&= 2\epsilon_k + \eta_k + s_k + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

and the theorem is satisfied.

Canonical inflation at next to leading order

- $\mathcal{O}(\epsilon)$ corrections to the previous result for $F_1 + F_2$

- Calculation of F_3 (naïvely divergent):

$$\frac{F_3}{P_\zeta(k)} = \left(\frac{\dot{\eta}_K}{H_K \eta_K}\right)\eta_K + \left(\frac{\dot{\eta}_K}{H_K \eta_K}\right)\eta_K \ln(-k\bar{\tau}) + \mathcal{O}(\bar{\tau}^2)$$

- The term from the field redefinition is evaluated at the late time $\bar{\tau}$ and not at horizon crossing:

$$\eta(\bar{\tau}) = \eta_K \left(1 - \frac{\dot{\eta}_K}{H_K \eta_K} \ln(-K\bar{\tau}) + \mathcal{O}(\epsilon^2)\right)$$

Useful check of the calculation: arbitrariness of the pivot scale K.

Conclusion

- Theoretical relevance of the consistency relation.

- "A convincing detection of a bispectrum signal in the squeezed limit (from primordial origin) would rule out all single field models of inflation" …

  up to models with interactions under the horizon (features).

- Checked by explicit calculations and different types of methods. Seery & Lidsey (05), Chen et al (06), Cheung et al (07), Ganc & Komatsu (10), Renaux-Petel (10)

- Subtleties at second order and use of an arbitrary pivot scale.