Ambiguities in second-order cosmological perturbations for non-canonical scalar fields

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Loop does not close

Maccallum & Taub '72

$$S = \int d^4x \sqrt{-g}\,[R + \mathcal{L}_{\mathrm{M}}] \qquad T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$$

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Vary the action

$$G_{\mu\nu} = T_{\mu\nu}$$

space-time symmetry

spatially homogeneous

equations of motion

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Vary the action	space-time symmetry
$G_{\mu\nu} = T_{\mu\nu}$	spatially homogeneous
space-time symmetry	Reduced action
spatially homogeneous	vary the reduced action
equations of motion	equations of motion

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Vary the action		space-time symmetry
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space-time symmetry		Reduced action
spatially homogeneous		vary the reduced action
equations of motion	$=$	equations of motion

for Class A group Eg. FRW, Bianchi III

Loop does not close

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- Vary the action
  - $G_{\mu\nu} = T_{\mu\nu}$
  - space-time symmetry
  - spatially homogeneous
  - equations of motion

$\neq$

- space-time symmetry
  - spatially homogeneous
  - Reduced action
  - vary the reduced action
  - equations of motion

for Class B group Eg. Bianchi II

Why is this result relevant for cosmological perturbations?

Several methods employed to study perturbations:

1. Einstein equations [Lifshitz 1946]
2. Covariant equations [Hawking 1966]
3. ADM equations [Bardeen 1980]
4. Action [Lukash 1980]

Why is this result relevant for cosmological perturbations?

These methods can be clubbed into two approaches:

$$S = \int d^4x \sqrt{-g}\,[R + \mathcal{L}_{\mathrm{M}}]$$

$G_{\mu\nu} = T_{\mu\nu}$

solve order-by-order

$G^{(0)}_{\mu\nu} = T^{(0)}_{\mu\nu}$

$\delta^{(1)} G_{\mu\nu} = \delta^{(1)} T_{\mu\nu}$

approach by Lifshitz, Bardeen

$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta^{(1)} g_{\mu\nu} + \cdots$

Reduced action

expand up to second or third order

approach by Lukash

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For canonical scalar field, these approaches agree up to second order

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For non-canonical scalar fields, unclear

Non-canonical scalars – Setup

Appignani etal '10

- Non-canonical scalar field Lagrangian:

$$\mathcal{L} = P(X, \phi) \quad \text{where} \quad 2\,X = \nabla^\alpha \phi \, \nabla_\alpha \phi$$
$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\mathbf{x}^2$$

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- Setup

1. freeze all the metric perturbations

   separate from the gauge ambiguities [Malik & Wands '10]

2. Look at the perturbations of the scalar field.

$$\phi(t,\mathbf{x}) = \phi_0(t) + \delta\phi(t,\mathbf{x}) + \delta^{(2)}\phi$$

Interested in second order perturbations

Perturbed tensor and ADM approach

Appignani etal '10

- Expand $T_{\mu\nu}$ to second order, $T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta^{(1)}T_{\mu\nu} + \delta^{(2)}T_{\mu\nu}$

$$\delta^{(2)}T_{00} = \left(P^{(0)}_{X} + 4\,P^{(0)}_{XX}\dot{\phi}_0^2 + P^{(0)}_{XXX}\dot{\phi}_0^4\right)\frac{\delta\dot{\phi}^2}{2} + \left(P^{(0)}_{X} - P^{(0)}_{XX}\dot{\phi}_0^2\right)\frac{\delta\phi_{,i}^2}{2\,a^2}$$
$$- \left(P^{(0)}_{\phi\phi} - P^{(0)}_{X\phi\phi}\dot{\phi}_0^2\right)\frac{\delta\phi^2}{2} + \left(P^{(0)}_{X\phi} + P^{(0)}_{XX\phi}\dot{\phi}_0^2\right)\dot{\phi}_0\,\delta\phi\,\delta\dot{\phi} + F[\delta^{(2)}\phi]$$

$$\delta^{(2)}T_{ii} = \frac{a^2}{2}\left(P^{(0)}_{X} + P^{(0)}_{XX}\dot{\phi}_0^2\right)\delta\dot{\phi}^2 + \frac{P^{(0)}_{X}}{2}\delta\phi_{,i}^2$$
$$+ \frac{a^2}{2}\left[2\,P^{(0)}_{X\phi}\dot{\phi}_0\,\delta\phi\,\delta\dot{\phi} + P^{(0)}_{\phi\phi}\delta\phi^2\right] + G[\delta^{(2)}\phi]$$

Perturbed tensor and ADM approach

Appignani etal '10

Salient features

- The ratio of coefficient of $\delta\phi_{,i}^2$ and $\delta\dot{\phi}^2$ can be related to the square of speed. For $\delta^{(2)}T_{00}$

$$c_0^2 = \frac{P_x^{(0)} - P_{xx}^{(0)}\dot{\phi}_0^2}{P_x^{(0)} + 4\,P_{xx}^{(0)}\dot{\phi}_0^2 + P_{xxx}^{(0)}\dot{\phi}_0^4}$$

Christopherson & Malik '09

- For the canonical scalar field $L = X - V(\phi)$,

$$\delta^{(2)}T_{00}^{(\mathrm{KG})} = \frac{\delta\dot{\phi}^2}{2} + \frac{\delta\phi_{,i}^2}{2\,a^2} + \frac{V_{\phi\phi}}{2}\,\delta\phi^2$$

positive definite $\qquad c_0^2 = 1$

- For any other Lagrangian, $\delta^{(2)}T_{00}$ is not positive; c_0^2 can be negative

Discussion for specific cases in the following slides

Symmetry reduced action

Appignani etal '10

- Canonical Hamiltonian corresponding to perturbed matter field action:

$$\delta^{(2)}\mathcal{H} = \frac{a^3}{2}\left[\left(P^{(0)}_{x} + P^{(0)}_{xx}\dot{\phi}^2_{0}\right)\delta\dot{\phi}^2_k + \left(\frac{k^2}{a^2}\,P^{(0)}_{x} - P^{(0)}_{\phi\phi}\right)\delta\phi^2_k\right]$$

Symmetry reduced action

Appignani etal '10

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Salient features

- Speed of perturbation

$$c_1^2 = \frac{P_{x}^{(0)}}{P_{x}^{(0)} + P_{xx}^{(0)}\dot{\phi}_{0}^2}$$

Garriga and Mukhanov '99

- $\delta^{(2)}\mathcal{H} = \delta^{(2)}T_{00}$ ONLY for Canonical scalar field.

  For non-canonical fields, the second order perturbed density from two approaches are not identical

- $c_0^2 = c_1^2$ for canonical scalar field

Examples

Appignani etal '10

Power-law k-inflation

Armendariz-Picon et al '99

- $P = f(\phi)\left(X^2 - X\right)$ $\qquad a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/(3\gamma)}$ $\qquad \gamma \in [0, \frac{2}{3}]$

- From background equations, we get

$$\dot{\phi}_0 = \sqrt{\frac{4 - 2\gamma}{4 - 3\gamma}} \qquad X^{(0)} = \frac{2 - \gamma}{4 - 3\gamma} \qquad X^{(0)} \in \left[\frac{1}{2}, \frac{2}{3}\right]$$

- Speed of perturbations from the two approaches are

$$c_0^2 = -\frac{2X^{(0)} + 1}{18X^{(0)} - 1} \implies c_0^2 < 0; \qquad c_1^2 = \frac{2X^{(0)} - 1}{6X^{(0)} - 1} \implies c_1^2 > 0$$

$\delta^{(2)}T_{00}$ has potential instability $\qquad \delta^{(2)}\mathcal{H}$ is stable

Examples

Appignani etal '10

	Lagrangian	Constraint from background	Constraint from II order
power-law k-inflation	$f(\phi)(X^2 - X)$	$X^{(0)} > \frac{1}{2}$	All values of $X^{(0)}$ are unstable
Tachyon	$-V(\phi)\sqrt{1-2X}$	$X^{(0)} < \frac{1}{2}$	$X^{(0)} < \frac{1}{4}$
DBI	$-\frac{1}{f(\phi)}\sqrt{1-2\,f(\phi)\,X}$ $\frac{1}{f(\phi)} - V(\phi)$	$X^{(0)} < \frac{1}{2f(\phi_0)}$	$X^{(0)} < \frac{1}{4f(\phi_0)}$

Conclusions

- At second order, the perturbed second order stress-tensor and canonical Hamiltonian are different.

  They are identical only for the canonical scalar field

- As in the case of gravity, the non-linear nature of non-canonical scalar fields is the key reason for this apparent discrepancy.

- Imperative to obtain f_{NL} by other approaches