

Ambiguities in second-order cosmological perturbations for non-canonical scalar fields

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space-time symmetry

spatially homogeneous

equations of motion

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Reduced action

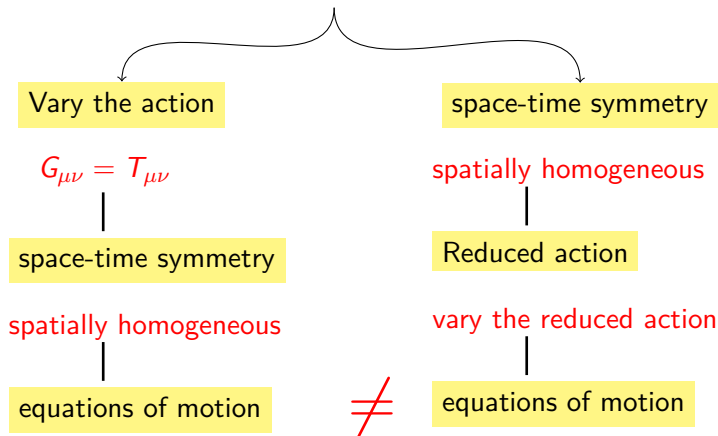
vary the reduced action

equations of motion

=

for Class A group Eg. FRW, Bianchi III

$$S = \int d^4x \sqrt{-g} [R + \mathcal{L}_M] \quad T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$$



for Class B group Eg. Bianchi II

Why is this result relevant for cosmological perturbations?

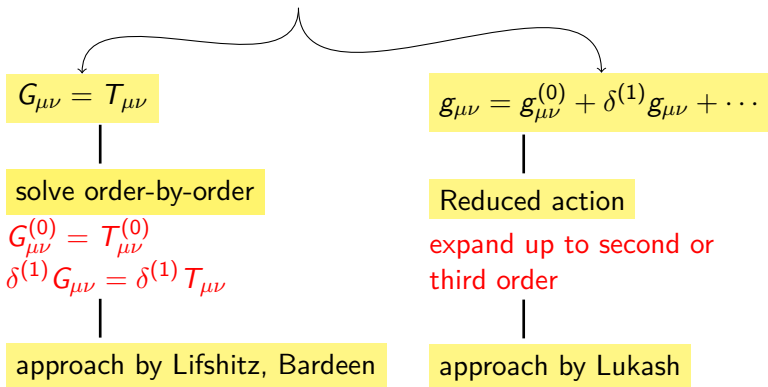
Several methods employed to study perturbations:

- | | |
|-----------------------|-----------------|
| ① Einstein equations | [Lifshitz 1946] |
| ② Covariant equations | [Hawking 1966] |
| ③ ADM equations | [Bardeen 1980] |
| ④ Action | [Lukash 1980] |

Why is this result relevant for cosmological perturbations?

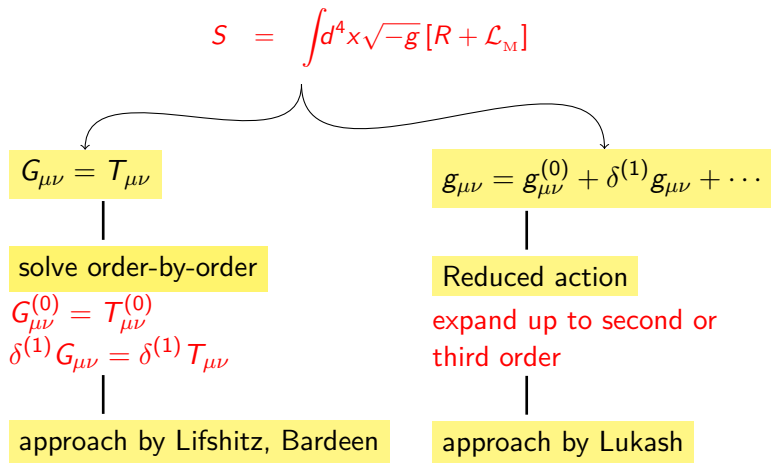
These methods can be clubbed into two approaches:

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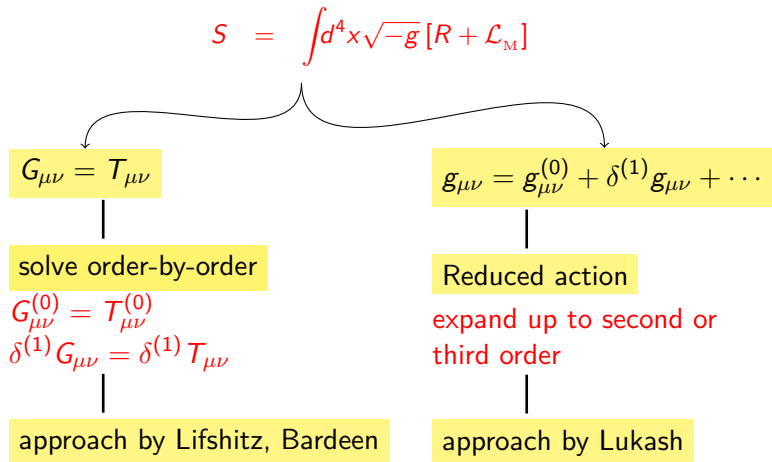
These methods can be clubbed into two approaches:



For canonical scalar field, these approaches agree up to second order

Why is this result relevant for cosmological perturbations?

These methods can be clubbed into two approaches:



For non-canonical scalar fields, unclear

- Non-canonical scalar field Lagrangian:

$$\mathcal{L} = P(X, \phi) \quad \text{where} \quad 2X = \nabla^\alpha \phi \nabla_\alpha \phi$$
$$ds^2 = dt^2 - a^2(t) d\mathbf{x}^2$$

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- Setup

- 1 freeze all the metric perturbations

separate from the gauge ambiguities

[Malik & Wands '10]

- 2 Look at the perturbations of the scalar field.

$$\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x}) + \delta^{(2)}\phi$$

Interested in second order perturbations

- Expand $T_{\mu\nu}$ to second order, $T_{\mu\nu} = T_{\mu\nu}^{(0)} + \delta^{(1)}T_{\mu\nu} + \delta^{(2)}T_{\mu\nu}$

$$\begin{aligned} \delta^{(2)}T_{00} = & \left(P_x^{(0)} + 4 P_{xx}^{(0)} \dot{\phi}_0^2 + P_{xxx}^{(0)} \dot{\phi}_0^4 \right) \frac{\delta \dot{\phi}^2}{2} + \left(P_x^{(0)} - P_{xx}^{(0)} \dot{\phi}_0^2 \right) \frac{\delta \phi_{,i}^2}{2 a^2} \\ & - \left(P_{\phi\phi}^{(0)} - P_{x\phi\phi}^{(0)} \dot{\phi}_0^2 \right) \frac{\delta \phi^2}{2} + \left(P_{x\phi}^{(0)} + P_{xx\phi}^{(0)} \dot{\phi}_0^2 \right) \dot{\phi}_0 \delta\phi \delta\dot{\phi} + F[\delta^{(2)}\phi] \end{aligned}$$

$$\begin{aligned} \delta^{(2)}T_{ii} = & \frac{a^2}{2} \left(P_x^{(0)} + P_{xx}^{(0)} \dot{\phi}_0^2 \right) \delta \dot{\phi}^2 + \frac{P_x^{(0)}}{2} \delta \phi_{,i}^2 \\ & + \frac{a^2}{2} \left[2 P_{x\phi}^{(0)} \dot{\phi}_0 \delta\phi \delta\dot{\phi} + P_{\phi\phi}^{(0)} \delta \phi^2 \right] + G[\delta^{(2)}\phi] \end{aligned}$$

Salient features

- The ratio of coefficient of $\delta\phi_{,i}^2$ and $\delta\dot{\phi}^2$ can be related to the square of speed. For $\delta^{(2)}T_{00}$

$$c_0^2 = \frac{P_x^{(0)} - P_{xx}^{(0)} \dot{\phi}_0^2}{P_x^{(0)} + 4 P_{xx}^{(0)} \dot{\phi}_0^2 + P_{xxx}^{(0)} \dot{\phi}_0^4}$$

Christopherson & Malik '09

- For the canonical scalar field $L = X - V(\phi)$,

$$\delta^{(2)}T_{00}^{(\text{KG})} = \frac{\delta\dot{\phi}^2}{2} + \frac{\delta\phi_{,i}^2}{2a^2} + \frac{V_{\phi\phi}}{2} \delta\phi^2 \quad \text{positive definite} \quad c_0^2 = 1$$

- For any other Lagrangian, $\delta^{(2)}T_{00}$ is **not positive**; c_0^2 can be negative

Discussion for specific cases in the following slides

- Canonical Hamiltonian corresponding to perturbed matter field action:

$$\delta^{(2)}\mathcal{H} = \frac{a^3}{2} \left[\left(P_x^{(0)} + P_{xx}^{(0)} \dot{\phi}_0^2 \right) \delta \dot{\phi}_k^2 + \left(\frac{k^2}{a^2} P_x^{(0)} - P_{\phi\phi}^{(0)} \right) \delta \phi_k^2 \right]$$

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Salient features

- Speed of perturbation

$$c_1^2 = \frac{P_x^{(0)}}{P_x^{(0)} + P_{xx}^{(0)} \dot{\phi}_0^2}$$

Garriga and Mukhanov '99

- $\delta^{(2)}\mathcal{H} = \delta^{(2)}T_{00}$ **ONLY** for Canonical scalar field.

For non-canonical fields, the second order perturbed density from two approaches are not identical

- $c_0^2 = c_1^2$ for canonical scalar field

Power-law k-inflation

Armendariz-Picon et al '99

- $P = f(\phi) (X^2 - X) \quad a(t) = a_0 \left(\frac{t}{t_0} \right)^{2/(3\gamma)} \quad \gamma \in [0, \frac{2}{3}]$

- From background equations, we get

$$\dot{\phi}_0 = \sqrt{\frac{4 - 2\gamma}{4 - 3\gamma}} \quad X^{(0)} = \frac{2 - \gamma}{4 - 3\gamma} \quad X^{(0)} \in \left[\frac{1}{2}, \frac{2}{3} \right]$$

- Speed of perturbations from the two approaches are

$$c_0^2 = -\frac{2X^{(0)} + 1}{18X^{(0)} - 1} \implies c_0^2 < 0; \quad c_1^2 = \frac{2X^{(0)} - 1}{6X^{(0)} - 1} \implies c_1^2 > 0$$

$\delta^{(2)}T_{00}$ has potential instability $\delta^{(2)}\mathcal{H}$ is stable

	Lagrangian	Constraint from background	Constraint from II order
power-law k-inflation	$f(\phi)(X^2 - X)$	$X^{(0)} > \frac{1}{2}$	All values of $X^{(0)}$ are unstable
Tachyon	$-V(\phi)\sqrt{1 - 2X}$	$X^{(0)} < \frac{1}{2}$	$X^{(0)} < \frac{1}{4}$
DBI	$-\frac{1}{f(\phi)}\sqrt{1 - 2f(\phi)X}$ $\frac{1}{f(\phi)} - V(\phi)$	$X^{(0)} < \frac{1}{2f(\phi_0)}$	$X^{(0)} < \frac{1}{4f(\phi_0)}$

Conclusions

- At second order, the perturbed second order stress-tensor and canonical Hamiltonian are different.

They are identical only for the canonical scalar field

- As in the case of gravity, the non-linear nature of non-canonical scalar fields is the key reason for this apparent discrepancy.
- Imperative to obtain f_{NL} by other approaches