Weighted Subsequence Sums in Finite Abelian Groups

By

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CERTIFICATE

This is to certify that the Ph.D. thesis entitled **"Weighted Subsequence Sums in Finite Abelian Groups"** submitted by Mohan N. Chintamani is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

Professor S. D. Adhikari

Thesis Supervisor

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Candidate

Date :

DECLARATION

This thesis is a presentation of my original research work. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussion.

The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

This work was done under the guidance of Professor S. D. Adhikari, at Harish-Chandra Research Institute, Allahabad.

Mohan Chintamani
Candidate

Date :
Place :

To My Family ...

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Mohan Chintamani
HRI

Abstract

A well-known result of Erdős-Ginzburg-Ziv [32] says that, given a sequence S of $2n-1$ integers, there is a subsequence S' of S with length n such that the sum of the terms of S' is zero modulo n.

Let G be an abelian group of order n, written additively. In [8] we have proved the following result :

Theorem 1.1 *Let G be a finite abelian group of order n, k a positive integer and 0 denote the identity element of G. Let $(w_1, w_2, \cdots, w_k)$ be a sequence of integers where each w_i is co-prime to n. Then, given a sequence $S\ :\ (x_1, x_2, \cdots, x_{k+r})$ of elements of G, where $1 \leq r \leq n-1$, if 0 is the most repeated element in S and*

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq 0,$$

for all permutations σ of $[k+r]$, we have

$$\left| \left\{ \sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \textit{ is a permutation of } [k+r] \right\} \right| \geq r+1.$$

Here, for a positive integer m, the notation $[m]$ is used for the set $\{1, 2, \cdots, m\}$. Given a finite non-empty subset A of integers, a sequence $(x_1, x_2, \cdots, x_l)$ of elements of G is said to be an A-weighted zero-sum sequence if $\sum_{i=1}^{l} a_i x_i = 0$, for some $a_i \in A$.

Abstract

We have proved some weighted generalization of the result of Bollobás and Leader.

Let $G \cong \mathbb{Z}/n\mathbb{Z}$, where n is a prime power or an odd integer greater than one and $A = (\mathbb{Z}/n\mathbb{Z})^*$ (the group of units modulo n). We derived a lower bound on the number of A-weighted n-sums of a sequence of elements of G which does not have an A-weighted zero-sum subsequence of length n. In what follows, $\varphi(m)$ is the number of integers t, $1 \leq t \leq m$ which are co-prime to m. Further, by $\Omega(m)$ (resp. $\omega(m)$) we denote the number of prime factors of m counted with multiplicity (resp. without multiplicity). We have proved the following results in [26].

Theorem 2.1 *Let p be any prime, $\alpha \geq r \geq 1$ and $A = (\mathbb{Z}/p^\alpha\mathbb{Z})^*$, the set of all units modulo p^α. Given a sequence $X = \{x_i\}_{i=1}^{p^\alpha+r}$ of integers, let*

$$S = \left\{ \sum_{i \in I} w_i x_i \pmod{p^\alpha} : I \subseteq [p^\alpha + r] \textit{ with } |I| = p^\alpha,\ w_i \in A \right\}.$$

If $0 \notin S$, then $|S| \geq p^{r+1} - p^r$.

Theorem 2.2 *Let n be an odd integer, $r \geq 1$ and $A = (\mathbb{Z}/n\mathbb{Z})^*$. Given a sequence $X = \{x_i\}_{i=1}^{n+r}$ of integers, let*

$$S = \left\{ \sum_{i \in I} w_i x_i \pmod{n} : I \subseteq [n + r] \textit{ with } |I| = n,\ w_i \in A \right\}$$

and $0 \notin S$. Then there exist primes $p_1, p_2, \ldots, p_{r+1}$ such that

$$|S| \geq \varphi(p_1)\varphi(p_2)\cdots\varphi(p_{r+1}) \textit{ with } p_1 p_2 \cdots p_{r+1} | n.$$

These are related to the results of F. Luca [53] and S. Griffiths [42], who independently confirmed a conjecture from [6].

A similar result is obtained, for general finite abelian group G and the weight set $A = \{1, -1\}$, in [7]. This is a weighted version of the result of Bollobás and Leader [20].

More precisely, we have proved the following result [7] :

Theorem 3.1 *Let G be a finite abelian group of order n and let it be of the form $G \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}$, where $1 < n_1 | n_2 | \cdots | n_r$. Let $A = \{1, -1\}$ and k be a natural number satisfying $k \geq 2^{r'-1} - 1 + \frac{r'}{2}$, where $r' = |\{i \in [r] : n_i \text{ is even}\}|$. Then, given a sequence $S = (x_1, x_2, \cdots, x_{n+k})$, with $x_i \in G$, if S has no A-weighted zero-sum subsequence of length n, then there are at least $2^{k+1} - \delta$ distinct A-weighted n-sums, where*

$$\delta = \begin{cases} 1 & \textit{if } 2 \mid n \\ 0 & \textit{otherwise.} \end{cases}$$

As a corollary, one obtains a result of Adhikari et. al. [6], giving the exact value of $E_A(G)$ for the cyclic case and unconditional bounds in many cases.

A result of Yuan and Zeng [71] on the existence of zero-smooth subsequence and the DeVos-Goddyn-Mohar Theorem [29] are some of the main ingredients of the proof of Theorem 3.1.

Synopsis

Thesis Title	**:**	Weighted Subsequence Sums in Finite Abelian Groups
Name	**:**	Mohan Chintamani
Supervisor	**:**	Prof. S. D. Adhikari
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This thesis contains some of my work in zero-sum problems done during my stay at Harish-Chandra Research Institute, under the supervision of Prof. S. D. Adhikari.

In 1961, Erdős-Ginzburg-Ziv [32] proved the following theorem :

Erdős-Ginzburg-Ziv (EGZ) Theorem : *If S is a sequence of $2n-1$ integers, then S contains an n length subsequence, sum of whose terms is 0 modulo n.*

The EGZ theorem is a corner-stone of the area of zero-sum problems in Combinatorial Number Theory. It continues to play a central role in the development of the area of zero-sum theorems and has been a subject of several generalizations.

Given a sequence $S : (a_1, a_2, \cdots, a_l)$ of elements of $\mathbb{Z}/n\mathbb{Z}$, one can naturally ask the following question :

How many distinct n-sums (by an n-sum we mean a sum of the terms of a subsequence of S of length n) can the sequence S have?

By taking $a_1 = a_2 = \cdots = a_l$, (in the group $\mathbb{Z}/n\mathbb{Z}$), one observes that the only n-sum is 0. Therefore, one asks the following :

Question. If 0 is not an n-sum, how many distinct n-sums a given sequence can have ?

The EGZ theorem says that there is no need to look at values of $l \geq 2n-1$ while dealing with the above question. A result of Bollobás and Leader [20] gave a lower bound on the number of n-sums of such a sequence in terms of the length of the sequence. Moreover, it gives a natural generalization of the EGZ theorem (as one obtains the EGZ theorem by taking $r = n-1$ in the following statement) :

Bollobás and Leader ([20]) : *Let G be an abelian group of order n and r be a positive integer with $r \leq n-1$. Let S denote the sequence $a_1, a_2, \cdots, a_{n+r}$ of $n+r$ elements of G. Then, if 0 is not an n-sum, the number of distinct n-sums of S is at least $r+1$.*

In an article [24], Y. Caro made the following conjecture :

Conjecture : (Y. Caro) *Let n, k be positive integers, $n \geq 2$. Let $(w_1, w_2, \cdots, w_k)$ be a sequence of integers such that $\sum_{i=1}^{k} w_i \equiv 0 \pmod{n}$. Given a sequence $S = (x_1, x_2, \cdots, x_{n+k-1})$ of $n+k-1$ integers (not necessarily distinct), there exists a permutation σ of $\{1, 2, \cdots, n+k-1\}$ such that*

$$\sum_{j=1}^{k} w_j x_{\sigma(j)} \equiv 0 \pmod{n}.$$

Clearly, taking $k = n$ and $w_i = 1,\ \forall i$, in above statement implies the EGZ theorem.

In [8], following the method of a simple proof of the above result of Bollobás and Leader as given by Yu [70], we obtained the following result, which generalizes the above theorem of Bollobás and Leader, and also supplies a new proof of a particular case (when $(w_i, n) = 1,\ \forall i$) of the above mentioned conjecture of Y. Caro (the first proof of the particular case of the conjecture was given by Hamidoune [47]).

Theorem 1.1 *Let G be a finite abelian group of order n and k a positive integer. Let $(w_1, w_2, \cdots, w_k)$ be a sequence of integers where each w_i is co-prime to n. Then, given a sequence $S : (x_1, x_2, \cdots, x_{k+r})$ of elements of G, where $1 \leq r \leq n-1$, if 0 is the most repeated element in S and*

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq 0,$$

for all permutations σ of $[k+r]$, we have

$$\left| \left\{ \sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } [k+r] \right\} \right| \geq r+1.$$

Here, for a positive integer m, by $[m]$ we denote the set $\{1, 2, \cdots, m\}$.

If in the above statement, instead of 0, x_1 happens to be the most repeated element, then applying the result on the sequence $(a_1, a_2, \cdots, a_{k+r})$, where $a_i = x_i - x_1,\ for\ all\ i = 1, 2, \cdots, k+r$, and observing that translation of a subset of G by an element does not change its cardinality, one obtains the following :

Corollary 1. *Let G be a finite abelian group of order n and k a positive integer. Let $(w_1, w_2, \cdots, w_k)$ be a sequence of integers where each w_i is co-prime to n. Then, given a sequence $S : (x_1, x_2, \cdots, x_{k+r})$ of elements of G, where $1 \leq r \leq n-1$, if*

x_1 is the most repeated element in S and

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq \left(\sum_{i=1}^{k} w_i\right) x_1,$$

for all permutations σ of $[k+r]$, we have

$$\left| \left\{ \sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } [k+r] \right\} \right| \geq r+1.$$

Further, taking $r = n-1$ in Corollary 1, one obtains the following result :

Corollary 2 (Hamidoune [47]). *Let G be a finite abelian group of order n and k a positive integer. Let $(w_1, w_2, \cdots, w_k)$ be a sequence of integers where each w_i is co-prime to n. Then, given a sequence $S : (x_1, x_2, \cdots, x_{k+n-1})$ of elements of G, if x_1 is the most repeated element in S, we have*

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} = \left(\sum_{i=1}^{k} w_i\right) x_1,$$

for some permutation σ of $[k+n-1]$.

The above result (Corollary 2) was proved by Hamidoune [47], confirming the above conjecture of Caro in a particular case when $(w_i, n) = 1$, $\forall i$. For further information regarding these results we refer to the paper of D. Grynkiewicz [43], where the conjecture of Caro has been established in full generality.

Shortly after the confirmation of Caro's conjecture, which introduced the idea of considering certain weighted subsequence sums, Adhikari and his collaborators (see [6], [12], [5]) initiated the study of a new kind of weighted zero-sum problems. Let G be a finite abelian group, written additively, of order n and A be a finite non-empty subset of integers. For a sequence $(x_1, x_2, \cdots, x_r)$ of elements of G, if there are $a_1, a_2, \cdots, a_r \in A$ such that $\sum_{i=1}^{r} a_i x_i = 0$, then the sequence is said to have 0 as an A-weighted sum, or simply to be an A-weighted zero-sum sequence. The *Davenport* constant with weight A, denoted by $D_A(G)$, is the least positive integer t such that any sequence of elements of G with length t has an A-weighted zero-sum subsequence of length ≥ 1. Similarly, $E_A(G)$ is defined to be the least positive integer t such that any sequence of elements of G with length t has an A-weighted zero-sum subsequence of length $n = |G|$.

Adhikari, Chen, Friedlander, Konyagin and Pappalardi [6] obtained the exact value of $D_A(G)$ and $E_A(G)$ with $A = \{1, -1\}$ and $G = \mathbb{Z}/n\mathbb{Z}$. In the same paper, it was conjectured that for $A = (\mathbb{Z}/n\mathbb{Z})^*$ (the group of units modulo n), one has

$$E_A(\mathbb{Z}/n\mathbb{Z}) = n + \Omega(n),$$

where $\Omega(n)$ denotes the number of prime factors of n counted with multiplicities. This conjecture was established independently by F. Luca [53] and S. Griffiths [42].

We obtained two results in [26], both of which are weighted versions of the theorem of Bollobás and Leader ([20]) and are related to the results of F. Luca and S. Griffiths :

Theorem 2.1 *Let p be any prime, $\alpha \geq r \geq 1$ and $A = (\mathbb{Z}/p^\alpha\mathbb{Z})^*$, the set of all units modulo p^α. Given a sequence $X = \{x_i\}_{i=1}^{p^\alpha+r}$ of integers, let*

$$S = \left\{ \sum_{i \in I} w_i x_i \pmod{p^\alpha} : I \subseteq [p^\alpha + r] \text{ with } |I| = p^\alpha,\ w_i \in A \right\}.$$

If $0 \notin S$, then $|S| \geq p^{r+1} - p^r$.

Theorem 2.2 *Let n be an odd integer, $r \geq 1$ and $A = (\mathbb{Z}/n\mathbb{Z})^*$. Given a sequence $X = \{x_i\}_{i=1}^{n+r}$ of integers, let*

$$S = \left\{ \sum_{i \in I} w_i x_i \pmod{n} : I \subseteq [n + r] \text{ with } |I| = n,\ w_i \in A \right\}$$

and $0 \notin S$. *Then there exist primes* $p_1, p_2, \ldots, p_{r+1}$ *such that*

$$|S| \geq \varphi(p_1)\varphi(p_2)\cdots\varphi(p_{r+1}) \text{ with } p_1p_2\cdots p_{r+1}|n.$$

As mentioned before, in [6] Adhikari, Chen, Friedlander, Konyagin and Pappalardi proved that,

$$\begin{aligned} D_A(\mathbb{Z}/n\mathbb{Z}) &= \lfloor \log_2 n \rfloor + 1, \\ E_A(\mathbb{Z}/n\mathbb{Z}) &= n + \lfloor \log_2 n \rfloor, \end{aligned}$$

where $A = \{1, -1\}$.

For a finite abelian group G of order n, in [7], we obtained a lower bound on the number of A-weighted n-sums of a sequence which does not have an A-weighted zero-sum subsequence of length n, with $A = \{1, -1\}$. In particular, this result gives an alternative proof of the main result from [6], giving the exact value of $E_A(G)$, for the cyclic case and unconditional bounds in many cases.

More precisely, we have proved the following result [7] :

Theorem 3.1 *Let G be a finite abelian group of order n and let it be of the form $G \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}$, where $1 < n_1|n_2|\cdots|n_r$. Let $A = \{1, -1\}$ and k be a natural number satisfying $k \geq 2^{r'-1} - 1 + \frac{r'}{2}$, where $r' = |\{i \in [r] : n_i \text{ is even}\}|$. Then, given a sequence $S = (x_1, x_2, \cdots, x_{n+k})$, with $x_i \in G$, if S has no A-weighted zero-sum subsequence of length n, then there are at least $2^{k+1} - \delta$ distinct A-weighted n-sums, where*

$$\delta = \begin{cases} 1 & \text{if } 2 \mid n \\ 0 & \text{otherwise.} \end{cases}$$

After this brief introduction to my thesis, I would like to indicate how this thesis is organized.

Each of the chapters is self contained. Chapter 1 is of introductory nature. It briefly discusses the Erdős-Ginzburg-Ziv Theorem and its generalization due to Bollobás and Leader [20]. After giving a proof of Scherk's Theorem [64] (which is a main tool we used to prove the Theorem 1.1), Chapter 2 presents our work from [8]. The third chapter begins by defining the generalized *Davenport* constant and the EGZ constant ($D_A(G)$, $E_A(G)$). It then continues with the discussion about the relation between these two constants. After proving some lemmas in Section 3.5, we give the proofs (as in [26]) of Theorems 2.1 and 2.2. Finally, Chapter 4 contains the proof of Theorem 3.1 (as in [7]), along with a short section on relating subsequence sums (of lengths ≥ 1) to $|G|$-sums of arbitrary sequence .

Contents

List of publications related to this thesis

- S. D. Adhikari, M. N. Chintamani, B. K. Moriya and P. Paul, *Weighted sums in finite abelian groups*, Unif. Distrib. Theory **3** (2008), 105–110.

- M. N. Chintamani, B. K. Moriya and P. Paul, *The Number of Weighted n-sums*, Int. J. Mod. Math., Vol **5** **(2)** 215–222 (2010).

- S. D. Adhikari, M. N. Chintamani, *Number of weighted subsequence sums with weights in* $\{1, -1\}$, Integers, **11** (2011), Paper A36.

Chapter 1

Zero Sum Problems: Early Development

1.1 Introduction

Let n be a positive integer. Given a sequence $a_1, a_2, \cdots, a_n$ of n integers, one can find a subsequence whose sum is a multiple of n. In other words, there exists a non-empty subset $I \subset \{1, 2, \cdots, n\}$ such that,

$$\sum_{i \in I} a_i \equiv 0 \pmod{n}. \tag{1.1}$$

Indeed, if one considers the following sums :

$$\begin{aligned} x_1 &= a_1, \\ x_2 &= a_1 + a_2, \\ &\vdots \\ x_n &= a_1 + a_2 + \cdots + a_n, \end{aligned}$$

then either an $x_i \equiv 0 \pmod{n}$ or by pigeonhole principle, at least two of the x_i's are equal modulo n, giving (1.1) for some I.

We observe that the above problem is framed in terms of the elements of the group $\mathbb{Z}/n\mathbb{Z}$ and is asking for a subsequence whose sum is 0, the identity element of $\mathbb{Z}/n\mathbb{Z}$. Naturally it has generalizations to arbitrary finite abelian groups. This leads to defining an important group invariant called the *Davenport Constant* of a group. In the subsequent chapters we study this constant and its important applications, apart from its generalizations.

In this chapter we consider the variation of the above problem where one puts some restrictions on the size of I. In particular, we have the following well-known theorem due to Erdős, Ginzburg and Ziv [32] :

Theorem 1.1 (EGZ Theorem) *Let n be a positive integer. Given a sequence $a_1, a_2, \cdots, a_{2n-1}$ of $2n-1$ integers, there exists a subsequence $a_{i_1}, a_{i_2}, \cdots, a_{i_n}$ of length n such that*

$$\sum_{j=1}^{n} a_{i_j} \equiv 0 \pmod{n}.$$

Henceforth we refer Theorem 1.1 as the EGZ theorem. The EGZ theorem is considered as a corner-stone of *zero-sum problems* in Combinatorial Number Theory. This theorem spurred the growth of the now developing field of zero-sum Ramsey Theory [23], [30], [65], and has been the subject of various generalizations [8], [13], [14], [19], [20], [25], [31], [33], [43], [47], [61], [60].

In this chapter, we survey this area, and try to summarize some of the early developments including the non-abelian case.

Including the paper of Erdős, Ginzburg and Ziv [32], there are several proofs of the EGZ theorem available in the literature (see [2], [17], [20], [56] for instance). We shall present two of these in the next section. In Section 1.3 we state a beautiful generalization of the EGZ theorem due to Bollobás and Leader [20]. Finally, in Section 1.4 we consider the non-abelian case of the EGZ theorem.

1.2 The EGZ Theorem

We begin by giving the first proof of Theorem 1.1. Following proof is due to Zimmerman which was presented by Alon and Dubiner in [17] (see also [2]).

Proof of Theorem 1.1 Firstly we prove the theorem for $n = p$, a prime. Consider S to be the following sum,

$$S = \sum_{\substack{I \subset [1,\ 2p-1], \\ |I| = p}} \left(\sum_{i \in I} a_i \right)^{p-1}.$$

We claim that

$$S \equiv 0 \pmod{p}. \tag{1.2}$$

We can write (after expanding) S as ,

$$S = \sum_{\substack{t_i \geq 0, \\ \sum t_i = p-1}} \left(C(t_1, \cdots, t_{2p-1}) \prod_{i \in [1,\ 2p-1]} a_i^{t_i} \right).$$

Consider a typical monomial of S, say $C(t_1, \cdots, t_{2p-1}) \prod_{i \in [1,\ 2p-1]} a_i^{t_i}$. Observe that the number of distinct terms a_i appearing in this monomial is at most $p-1$. The p-subsets I of $[1,\ 2p-1]$ that contributes to its coefficient are precisely those containing distinct indices of a_i's appearing in this monomial. Further, each such I contributes equally to the coefficient. If j is the number of distinct terms appearing, the number of such I's is

$$\binom{2p-1-j}{p-j} \equiv 0 \pmod{p}.$$

Thus $C(t_1, \cdots, t_{2p-1}) \equiv 0 \pmod{p}$. Since this is true for each monomial occurring in S, we have proved the claim (1.2).

Now, if possible assume that

$$\sum_{i \in I} a_i \not\equiv 0 \pmod{p},$$

for each $I \subset [1,\ 2p-1],\ |I| = p$. By Fermat's little theorem, for each $I \subset [1,\ 2p-1],\ |I| = p$ we have,

$$\left(\sum_{i \in I} a_i\right)^{p-1} \equiv 1 \pmod{p}.$$

Since the number of p-subsets I of $[1,\ 2p-1]$ is $\binom{2p-1}{p}$, we get

$$\begin{aligned} S &\equiv \binom{2p-1}{p} \pmod{p} \\ &\equiv 1 \pmod{p}. \end{aligned}$$

This contradicts to (1.2). This establishes Theorem 1.1 for $n = p$ prime.

Now we proceed to prove the theorem for general n, by induction on number of prime factors (counted with multiplicity) of n. For the case $n = 1$, there is nothing to prove and the result is true in the case when n is a prime.

Now, let $n > 1$ is not a prime and assume that the theorem holds true for all integers having less number of prime factors than that of n. We write $n = mp$ where p is a prime and $m > 1$.

By our assumption, each subsequence of $2p-1$ members of the sequence $a_1, a_2, \cdots, a_{2n-1}$ has a subsequence of p elements whose sum is 0 modulo p. From the original sequence we go on repeatedly omitting such subsequences of p elements having sum equal to 0 modulo p. Even after $2m-2$ such sequences are omitted, we are left with $2pm-1-(2m-2)p = 2p-1$ elements and we can have at least one more subsequence of p elements with the property that sum of its elements is equal to 0 modulo p.

Thus, we have found $2m-1$ pairwise disjoint subsets $I_1, I_2, \cdots, I_{2m-1}$ of $\{1, 2, \cdots, 2mp-1\}$ with $|I_i| = p$ and $\sum_{j \in I_i} a_j \equiv 0 (\text{mod } p)$ for each i. Consider the sequence $b_1, b_2, \cdots, b_{2m-1}$, where for each $i \in \{1, 2, \cdots, 2m-1\}$,

$$b_i = \frac{1}{p} \sum_{j \in I_i} a_j.$$

By the induction hypothesis, there exist distinct indices $i_1, i_2, \cdots, i_m$ such that $\sum_{j=1}^{m} b_{i_j} \equiv 0 \pmod{m}$.

Therefore, we have,

$$\sum_{j=1}^{m} \sum_{i \in I_j} a_i \equiv 0 \pmod{pm}.$$

Thus the result is true for n. Hence we are through by induction. $\square$.

We observe (by above arguments) that it is enough to prove the EGZ theorem for prime case. For our second proof of the EGZ theorem, we shall need a generalized version of *Cauchy-Davenport Theorem* ([22], [27], can also look into [54] or [56] for instance).

For a finite abelian group G, written additively, and A, B two non-empty subsets of G, we define the *sumset* $A + B$ of A and B as :

$$A + B = \{a + b : a \in A, b \in B\}.$$

Theorem 1.2 (Cauchy-Davenport) *Let $A_1, A_2, \cdots, A_h$ be h non-empty subsets of $\mathbb{Z}/p\mathbb{Z}$, where p is a prime. Then*

$$\left|\sum_{i=1}^{h} A_i\right| \geq \min\left(p, \sum_{i=1}^{h} |A_i| - h + 1\right).$$

Let $n = p$ be a prime and consider the least non-negative representatives modulo p of the given elements a_i's. i.e $0 \leq a_i \leq p-1$. If necessary, by rearranging we assume that

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_{2p-1} \leq p-1.$$

Further, we can assume that

$$a_j \neq a_{j+p-1}, \quad \text{for } j = 1, 2, \cdots, p-1.$$

For, otherwise we have $a_j = a_{j+1} = \cdots = a_{j+p-1}$, for some j, $1 \leq j \leq p-1$ and the result holds trivially.

Now, applying Theorem 1.2 on the sets

$$A_j := \{a_j, a_{j+p-1}\}, \text{ for } j = 1, \cdots, p-1,$$

we get

$$\begin{aligned} \left|\sum_{j=1}^{p-1} A_j\right| &\geq \min\left(p, \sum_{j=1}^{p-1} |A_j| - (p-1) + 1\right) \\ &= p. \end{aligned}$$

Thus,

$$\sum_{j=1}^{p-1} A_j = \mathbb{Z}/p\mathbb{Z}.$$

In particular,

$$-a_{2p-1} \in \sum_{j=1}^{p-1} A_j$$

and hence once again we have established the EGZ theorem for the case when n is a prime. $\square$

Remark 1.1. Results like the EGZ theorem (like many other zero-sum results) can also find their place in a larger class of results in combinatorics; for example, in (as mentioned before) zero-sum Ramsey Theory [23], [30], [65]. These are termed as *Ramsey-type theorems* in combinatorics. One can refer to [2] and references given therin, for similar results.

Remark 1.2. We observe that in Theorem 1.1, the number $2n-1$ is the smallest positive integer for which the theorem holds. Define $E(n)$ to be the smallest positive integer t such that given a sequence $a_1, a_2, \cdots, a_t$ of not necessarily distinct integers, there exists a set $I \subset \{1, 2, \cdots, t\}$ with $|I| = n$ such that $\sum_{i \in I} a_i \equiv 0 \pmod{n}$. Theorem 1.1 implies that $E(n) \leq 2n-1$. On the other hand, consider the sequence

$$\underbrace{0, 0, \cdots, 0}_{\text{n−1 times}}, \underbrace{1, 1, \cdots, 1}_{\text{n−1 times}}$$

of $2n-2$ integers modulo n. Clearly, this sequence does not have any subsequence of n elements sum of whose elements is 0 modulo n. Thus $E(n) \geq 2n-2+1 = 2n-1$. Hence we have $E(n) = 2n-1$. The constant $E(n)$ and its generalization will be studied in the subsequent chapters (see Sections 3.2 & 3.4 of Chapter 3).

1.3 A Generalization of EGZ Theorem

In this section we mention one of the beautiful generalization of the EGZ theorem, due to Bollobás and Leader [20]. Let G be a finite abelian group. In Theorem 1.1, the authors study only sequences of length $2n-1$ and subsequences of length n, where $G = \mathbb{Z}/n\mathbb{Z}$ and $|G| = n$. Perhaps the most obvious attempt to generalization would be to consider different length sequences and subsequences of elements of G. Indeed, Bollobás and Leader in [20] found a nice generalization of Theorem 1.1 by considering subsequences of length $|G|$ from an arbitrary sequence.

For a sequence $x_1, x_2, \cdots, x_m$ of elements of G and $1 \leq t \leq m$, by an t-sum we mean a sum $x_{i_1} + \cdots + x_{i_t}$ of a subsequence of length t.

The result of Bollobás and Leader [20] mentioned above is the following :

Theorem 1.3 (Bollobás and Leader) *Let G be an abelian group of order n and r be a positive integer. Let S denote the sequence $a_1, a_2, \cdots, a_{n+r}$ of $n+r$ not necessarily distinct elements of G. Then, if 0 is not an n-sum, the number of distinct n-sums of S is at least $r+1$.*

By taking $r = n-1$, one obtains the EGZ theorem from the above result. In fact, we have a generalization of it to finite abelian groups, as follows :

Theorem 1.4 *Let G be an abelian group of order n. Given any sequence $x_1, x_2, \cdots, x_{2n-1}$ of $2n-1$ elements of G, there exists a subsequence of n elements whose sum is the identity element 0 of G.*

However, it is not difficult to see that using structure theorem for finite abelian groups and arguing by induction on the rank of G, one can derive Theorem 1.4 from the EGZ theorem itself.

Remark 1.3. We define the constant $E(G)$ for a finite abelian group G to be the least integer t with the property that given any sequence of elements of G of length t, it has a subsequence of length $|G|$ whose elements sums up to the 0 (the identity element of G). In particular, in Remark 1.2, using Theorem 1.1, we observed that

$$E(\mathbb{Z}/n\mathbb{Z}) = E(n) = 2n-1.$$

From Theorem 1.4, it follows that $E(G) \leq 2|G|-1$, for any finite abelian group G. However, for a non-cyclic abelian group G of order n, $E(G)$ need not be equal to $2n-1$. In this direction, a result of Alon, Bialostocki and Caro [16] says that for a non-cyclic abelian group G of order n, $E(G) \leq \frac{3n}{2}$ and the bound $\frac{3n}{2}$ is realized only by groups of the form $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2m\mathbb{Z}$. Subsequently, Y. Caro [23] showed that if a non-cyclic abelian group G of order n is not of the form $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2m\mathbb{Z}$, then $E(G) \leq \frac{4n}{3}+1$ and this bound is realized only by groups of the form $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3m\mathbb{Z}$.

Further generalizations of the same nature have been obtained by Ordaz and Quiroz [62]. One of the generalizations of Theorem 1.3 obtained by Hamidoune [48] will be discussed in Chapter 3.

Now, we give a brief sketch of the proof of Theorem 1.3.

Proof of Theorem 1.3 (Sketch) :

We may clearly assume that $r < n$. For an abelian group G and elements $b_1, b_2, \cdots, b_r \in G$, we denote by $S(b_1, b_2, \cdots, b_r)$ the set,

$$S(b_1, b_2, \cdots, b_r) = \left\{ \sum_{i \in I} b_i : I \subset \{1, 2, \cdots, r\} \right\},$$

of all 2^r possible sums (including the empty sum, which we define to be 0). We begin by considering that under what conditions the set $S(b_1, b_2, \cdots, b_r)$ is large. For example, if order of each of the b_i's is co-prime to $n(=|G|)$, then it follows by induction on r that $|S(b_1, b_2, \cdots, b_r)| \geq r+1$. The relevance for the proof is that, for the given sequence, the set

$$\{a_1, a_2\} + \{a_3, a_4\} + \cdots + \{a_{2r-1}, a_{2r}\} + a_{2r+1} + \cdots + a_{n+r}$$

which is a subset of the set of all n-sums of the given sequence, has precisely $|S(a_1 - a_2, a_3 - a_4, \cdots, a_{2r-1} - a_{2r})|$ elements. Bollobás and Leader proved the following lemma, which simply asserts that $|S(b_1, b_2, \cdots, b_r)| \geq r+1$ unless the b_i's are very densely concentrated on some (proper) subgroup of G.

Lemma 1.1 *Let G be a finite abelian group, and let $b_1, b_2, \cdots, b_r$ be elements of G. Suppose that, for each subgroup H of G, the number of b_i's belonging to H is less than $r + 1 - r/|H|$. Then $|S(b_1, b_2, \cdots, b_r)| \geq r + 1$.*

Now, we try to find disjoint pairs of the a_i's whose differences b_i are not too highly concentrated in any subgroup of G; in other words, we can apply the above lemma. If such pairs exist then using the above arguments, we get the result. Otherwise, too many of the a_i's lie in some coset of some subgroup H, and then one could apply the induction, by looking at H, thus completing the proof. $\square$.

The original proof [20] of Theorem 1.3 is somewhat difficult and complicated. A simple combinatorial proof of the above theorem has been given by Yu [70] using the following result of Scherk [64] (see also [51] and Theorem 15′ of Chap. 1 in [46]).

Theorem 1.5 (P. Scherk) *Let A and B be two subsets of an abelian group G of order n. Suppose $0 \in A \cap B$ and that the only solution of the equation $a + b = 0,\ a \in A, b \in B$ is $a = 0 = b$. Then,*

$$|A + B| \geq |A| + |B| - 1.$$

This result of Scherk is an analogue of the Cauchy-Davenport Theorem (Theorem 1.2).

Remark 1.4. It is not difficult to observe that with the assumption in the Theorem 1.5, the number $|A|+|B|-1$ cannot exceed $|G|$. Indeed, letting $-A = \{-a : a \in A\}$, we clearly have $|A| = |(-A)|$. Then assuming $|A| + |B| - 1 > |G|$, we get

$$\begin{aligned} |G| &\geq |(-A) \cup B| \\ &= |(-A)| + |B| - |(-A) \cap B| \\ &> |G| + 1 - |(-A) \cap B|, \\ \Rightarrow \quad |(-A) \cap B| &> 1. \end{aligned}$$

Thus, for some $x \in A$, $y \in B$, with $x \neq 0 \neq y$ we have $x + y = 0$. Contradicting to the hypothesis.

Remark 1.5. In [8], following the method of a simple proof of the above result of Bollobás and Leader as given by Yu [70], we obtained a result, which will be presented in the next chapter. This result gives a generalization of the EGZ theorem, and imply a result of Hamidoune [47], which had confirmed a conjecture (Conjecture 2.1 of Chapter 2) of Y. Caro [24] (see also [43], for instance) in a special case. For further information regarding these results, we refer to the paper of Grynkiewicz [43], where the above mentioned conjecture of Y. Caro has been established in full generality. We will discuss this in Chapter 2.

As Theorem 1.5 is a tool we used in [8], we shall give one of its proof in Chapter 2.

We shall end this section by giving a result due to W. D. Gao which, in a particular case (when n is prime), gives more information than the EGZ theorem.

Theorem 1.6 (Gao) *For a cyclic group G of prime order p and any element a of G, given an arbitrary sequence $S : \{g_1, g_2, \cdots, g_{2p-1}\}$ of $2p - 1$ elements of G, if $r(S, a)$ denotes the number of the subsequences of S of length exactly p whose sum is a, then*

$$r(S, a) \equiv \begin{cases} 1 \pmod{p} \text{ if } a = 0 \\ 0 \pmod{p} \text{ otherwise.} \end{cases}$$

1.4 The Non-Abelian Case

As observed in Section 1.3, the EGZ theorem holds true for any finite abelian group G, viz Theorem 1.4. Its natural to ask the validity of Theorem 1.4 for arbitrary finite groups, not necessarily abelian.

For a finite solvable group G, we have the following result [66] :

Theorem 1.7 *Let G be a finite solvable group (written additively) of order n. Then given any sequence $x_1, x_2, \cdots, x_{2n-1}$ of $2n-1$ elements of G, there exist n distinct indices $i_1, i_2, \cdots, i_n$ such that*

$$x_{i_1} + x_{i_2} + \cdots + x_{i_n} = 0.$$

As mentioned in [66], one can obtain the above result again by using Theorem 1.1 and arguments used to derive general case of it from the 'prime' case, by induction on the length k of a minimal abelian series of subgroups (a series $(0) = G_0 \subset G_1 \subset \cdots \subset G_k = G$ of subgroups of G is said to be abelian series if each G_i is a normal subgroup of G_{i+1} and G_{i+1}/G_i is abelian) for G,

J. E. Olson in [60], generalizing a result of H. B. Mann [55], showed that above result holds for any finite group G (not necessarily solvable).

We note that above result of Olson [60] (like Theorem 1.7) does not guarantee a *subsequence* of given sequence sum of whose elements is 0, rather a permutation of a subsequence of length n will do so. However, it is conjectured [60] that there must be a subsequence summing up to 0. In other words :

Conjecture 5.1 (Olson) *Let G be a finite group of order n. Given any sequence $x_1, x_2, \cdots, x_{2n-1}$ of $2n-1$ elements of G, there exist n indices $i_1, i_2, \cdots, i_n$ with $1 \leq i_1 < i_2 < \cdots < i_n \leq 2n-1$ such that*

$$x_{i_1} + x_{i_2} + \cdots + x_{i_n} = 0.$$

This is not known even for solvable groups. However, by pigeonhole principle, one can see that, if we take the length of the sequence to be $n(n-1)+1$ (instead of $2n-1$), then at least one element is repeated n times and the assertion of Conjecture 5.1 holds.

Chapter 2

A Weighted Erdős-Ginzburg-Ziv Theorem

2.1 Introduction

In this chapter, as mentioned in Remark 1.5 (of Chapter 1), we shall present our results as in [8].

Let G be a finite abelian group, written additively. In what follows, we use the notation $[n]$ to denote the set $\{1, 2, \cdots, n\}$. Also for $m \in \mathbb{Z}$, $A \subset G$ by mA we denote the set $\{ma : a \in A\}$. We recall (Section 1.3 of Chapter 1) that, for a sequence $x_1, x_2, \cdots, x_m$ of elements of G and $1 \leq t \leq m$, by an t-sum we mean a sum $x_{i_1} + \cdots + x_{i_t}$ of a subsequence of length t. Further, if $W : (w_1, w_2, \cdots, w_t)$ is a sequence of integers, by a t-sum with weights from W we mean a sum of the form $w_1 x_{i_1} + \cdots + w_t x_{i_t}$.

In an article [24], Y. Caro made the following conjecture :

Conjecture 2.1(Y. Caro) Let n, k be positive integers, $n \geq 2$. Let

$(w_1, w_2, \cdots, w_k)$ be a sequence of integers such that $\sum_{i=1}^{k} w_i \equiv 0 \pmod{n}$. Given a sequence $S : x_1, x_2, \cdots, x_{n+k-1}$ of $n+k-1$ integers, there exists a permutation σ of $\{1, 2, \cdots, n+k-1\}$ such that

$$\sum_{j=1}^{k} w_j x_{\sigma(j)} \equiv 0 \pmod{n}.$$

Clearly, taking $k = n$ and $w_i = 1$, $\forall i$, in the above statement implies the EGZ theorem (Theorem 1.1).

N. Alon proved Conjecture 2.1 in the case $k = n$, with n prime. In 1996 Hamidoune [47] proved the above conjecture in a particular case where w_i's are co-prime to n, but in a more general abelian group setting. Until 2004 no further progress was made on this conjecture. W. Gao and X. Jin [37] established the above conjecture in the case when n is a square of a prime. Finally, D. Grynkiewicz [43] settled Conjecture 2.1 completely.

Remark 2.1. We can not expect $(x_{\sigma(j)})$ (where σ is as in the statement of the Conjecture 2.1) to be a subsequence (in order) of S. For, we take $n = k$ and the following sequences (see [43]):

$$(w_1, w_2, \cdots, w_n) = (\underbrace{1, 1, \cdots, 1}_{\text{n}-2 \text{ times}}, 0, 2),$$

$$(x_1, x_2, \cdots, x_{n+n-1}) = (-1, \underbrace{0, \cdots, 0}_{\text{n}-1 \text{ times}}, \underbrace{1, \cdots, 1}_{\text{n}-1 \text{ times}}).$$

In [8], we proved the following weighted generalization of Theorem 1.3 of Bollobás and Leader ([20]).

Theorem 2.1 *Let G be a finite abelian group of order n and k a positive integer. Let $(w_1, w_2, \cdots, w_k)$ be a sequence of integers where each w_i is co-prime to n. Then,* *given a sequence* $S : (x_1, x_2, \cdots, x_{k+r})$ *of elements of* G*, where* $1 \leq r \leq n-1$*, if* 0 *is the most repeated element in* S *and*

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq 0, \tag{2.1}$$

for all permutations σ *of* $[k+r]$*, we have*

$$\left| \left\{ \sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \textit{ is a permutation of } [k+r] \right\} \right| \geq r+1.$$

We shall give the proof of the above theorem in the next section. Now, we give some of the corollaries. Its trivial to observe that for any $A \subset G$, $x \in G$ we have $|A - x| = |A|$. Thus, we have the following corollary, by applying Theorem 2.1 to the sequence $(x_1 - x_1, x_2 - x_1, \cdots, x_{k+r} - x_1)$:

Corollary 1. *Let* G *be a finite abelian group of order* n *and* k *a positive integer. Let* $(w_1, w_2, \cdots, w_k)$ *be a sequence of integers where each* w_i *is co-prime to* n*. Then, given a sequence* $S : (x_1, x_2, \cdots, x_{k+r})$ *of elements of* G*, where* $1 \leq r \leq n-1$*, if* x_1 *is the most repeated element in* S *and*

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq \left(\sum_{i=1}^{k} w_i \right) x_1,$$

for all permutations σ *of* $[k+r]$*, we have*

$$\left| \left\{ \sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \textit{ is a permutation of } [k+r] \right\} \right| \geq r+1.$$

In Corollary 1, taking $r = n-2$ and letting

$$\alpha = \left(\sum_{i=1}^{k} w_i \right) x_1,$$

if

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq \alpha,$$

for all permutations σ of $[k+n-2]$, then for each element $\beta \in G \setminus \{\alpha\}$ there exists a permutation τ of $[k+n-2]$ such that

$$\beta = \sum_{i=1}^{k} w_i x_{\tau(i)}.$$

Similarly, taking $r = n-1$ we obtain the following corollary :

Corollary 2 (Hamidoune [47]). *Let G be a finite abelian group of order n and k a positive integer. Let $(w_1, w_2, \cdots, w_k)$ be a sequence of integers where each w_i is co-prime to n. Then, given a sequence $S\ :\ (x_1, x_2, \cdots, x_{k+n-1})$ of elements of G, if x_1 is the most repeated element in S, we have*

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} = \left(\sum_{i=1}^{k} w_i\right) x_1,$$

for some permutation σ of $[k+n-1]$. In particular, if w_i's satisfy the equation

$$\sum_{i=1}^{k} w_i \equiv 0 \pmod{n},$$

then for some permutation σ of $[k+n-1]$, we have,

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} = 0.$$

Remark 2.2. Taking $k = n$ and $w_i = 1$ for each i, the set

$$\left\{\sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } [k+r]\right\}$$

is clearly a subset of the set of all n-sums of given sequence; thus Theorem 2.1 imply Theorem 1.3 of Bollobás and Leader ([20]). Clearly, each of the above corollaries gives a generalization of the EGZ theorem (also of Theorem 1.4), simply by taking $k = n$, $r = n - 1$ and $w_i = 1$, for $i = 1, 2, \cdots, k$. Moreover, Corollary 2 implies a particular case (when $(w_i, n) = 1 \ \forall\ i$) of Conjecture 2.1.

2.2 Proof of Theorem 1.5 (Scherk's Theorem)

In this section we shall present a proof of Theorem 1.5 as given in [64].

Recall that we have to show the following inequality,

$$|A + B| \geq |A| + |B| - 1. \tag{2.2}$$

We argue by induction on $|B|$.

For $|B| = 1$, theorem is obvious. Assume the result for $|B| \leq m - 1$, and let $|B| = m > 1$.

Let $b_0 \in B$, $b_0 \neq 0$. Then from the hypothesis on A, B we have $0 \notin A + b_0$. Since $|A| = |A + b_0|$ and $0 \in A$, there exists $a_0 \in A$ such that $a_0 + b_0 \notin A$.

Define the following sets :

$$\begin{aligned} B_1 &= \{y \in B : a_0 + y \notin A\}, \\ A_1 &= \{a_0 + y : y \in B_1\}. \end{aligned}$$

Since $0 \notin B_1$ and $b_0 \in B_1$ we get

$$0 < |A_1| = |B_1| < |B|. \tag{2.3}$$

Thus, for the sets

$$\begin{aligned} A_2 &:= A \cup A_1, \\ \text{and} \quad B_2 &:= B \setminus B_1, \end{aligned}$$

we have from (2.3),

$$0 \in B_2 \subsetneq B \ ; \quad 0 \in A \subset A_2. \tag{2.4}$$

Further,

$$\begin{aligned} |A_2| + |B_2| &= |A| + |A_1| + |B_2| \\ &= |A| + |B_1| + |B_2| \\ &= |A| + |B|. \end{aligned}$$

By definition of sets A_2 and B_2, $A_2 + B_2 \subset A + B$. Suppose that $x + y = 0$ holds for some $x \in A_2 \setminus \{0\}$, $y \in B_2$. Then by hypothesis on A and B, we have $x \in A_1$. So there exists $y' \in B_1$ such that $x = a_0 + y'$. Then, using definition of B_2,

$$0 = x + y = (a_0 + y) + y' \in A + B.$$

This contradicts the hypothesis. It follows that the only solution of $x + y = 0$, with $x \in A_2$, $y \in B_2$ is $x = 0 = y$. Hence in view of (2.4), we can apply induction to the sets A_2, B_2 and get

$$\begin{aligned} |A + B| &\geq |A_2 + B_2| \\ &\geq |A_2| + |B_2| - 1 \\ &= |A| + |B| - 1. \end{aligned}$$

Hence we are through, by induction. $\square$.

2.3 Proof of Theorem 2.1

We have given a sequence $S : (x_1, x_2, \cdots, x_{k+r})$. Let $L = \{i : x_i = 0\}$ and $|L| = l$. By the hypothesis (2.1) on S we have $1 \leq l \leq k - 1$.

Let $I \subset [k + r] \setminus L$ be such that $|I| = s$ is maximal subject to the conditions

- $s \leq k-1$ and
- there is an injective map $f : I \to [k]$ such that

$$\sum_{i \in I} w_{f(i)} x_i = 0.$$

It is possible that I is empty.

We note that

$$l + s \leq k - 1. \tag{2.5}$$

For, if $l + s \geq k$ then $l \geq k - s (\geq 1)$. By selecting $k - s$ distinct indices $i_1, i_2, \cdots, i_{k-s} \in L$ we have

$$w_{j_1} x_{i_1} + w_{j_2} x_{i_2} + \cdots + w_{j_{k-s}} x_{i_{k-s}} + \sum_{i \in I} w_{f(i)} x_i = 0,$$

where $\{j_1, j_2, \cdots, j_{k-s}\} = [k] \setminus \{f(i) : i \in I\}$. This is a contradiction to the hypothesis (2.1) on S.

Now, from (2.5) we have,

$$\begin{aligned} |[k+r] \setminus (L \cup I)| &= k + r - (l + s) \\ &\geq k + r - (k - 1) \\ &= r + 1. \end{aligned}$$

Therefore, there is a subset $J \subset [k+r] \setminus (L \cup I)$ such that $|J| = r$. Let h be the maximum number of repetition of an element in the sequence $X = (x_j : j \in J)$. Recall that we have 0 is the most repeated element in S and the definition of L, we have $h \leq l$. Thus from (2.5), we get,

$$h + s \leq l + s \leq k - 1. \tag{2.6}$$

Let $X = X_1 \cup X_2 \cup \cdots \cup X_h$ be a partition of the sequence X into h non-empty subsets, that is, in a particular X_i no element is repeated (X_i's need not be disjoint). More precisely this is done as follows : Let x be an element in X which is repeated h times (existence of such an element follows by definition of h). We put x in each X_i. Any other element, say y, occurring in X appears $m \leq h$ times and we put y in X_i, $1 \leq i \leq m$. (Thus X_i's can be treated as sets). We continue this process until all the elements of X are exhausted.

So,

$$|X_1| + |X_2| + \cdots + |X_h| = \text{ length of X} = r. \tag{2.7}$$

From (2.6), $h < k - s$. Let $w'_1, w'_2, \cdots, w'_{k-s}$ be the subsequence $(w_i : i \in I')$ where ,

$$I' = \{j \in [k] : j \neq f(i) \ \forall \ i \in I\}.$$

We claim that

$$0 \notin w'_1 X_1 + w'_2 X_2 + \cdots + w'_j X_j, \tag{2.8}$$

for each j, $1 \leq j \leq h$.

If possible, suppose that

$$0 = w'_1 x_{i_1} + w'_2 x_{i_2} + \cdots + w'_j x_{i_j},$$

for some $x_{i_t} \in X_t$, $t = 1, 2, \cdots, j$. Then we have,

$$w'_1 x_{i_1} + w'_2 x_{i_2} + \cdots + w'_j x_{i_j} + \sum_{i \in I} w_{f(i)} x_i = 0. \tag{2.9}$$

By (2.6), it follows that

$$\begin{aligned} |I| &< |I \cup \{i_1, i_2, \cdots, i_j\}| \\ &= s + j \\ &\leq h + s \\ &\leq k - 1. \end{aligned}$$

Thus, in view of (2.9), we are led to a contradiction to the maximality of I, by replacing I with $I \cup \{i_1, i_2, \cdots, i_j\}$ ($\subset [k+r] \setminus L$).

This establishes the claim (2.8).

Writting $X'_t = X_t \cup \{0\}$, for $t = 1, 2, \cdots, h$, and observing that $|cX'_t| = |X'_t|$, for any integer c co-prime to n, from (2.7), we have

$$\sum_{i=1}^{h} |w'_i X'_i| = r + h.$$

Therefore, by repeated application of Theorem 1.5(of Chapter 1) of Scherk, we get (note that by (2.8) hypothesis of Theorem 1.5 is satisfied each time),

$$\begin{aligned} \left| \sum_{i=1}^{h} w'_i X'_i \right| &\geq \min\left\{ n,\ \sum_{i=1}^{h} w'_i X'_i - (h-1) \right\} \\ &= \min\{n,\ r+1\} \\ &= r+1. \end{aligned}$$

Therefore, if we append a subsequence of h zeros from $(x_i : i \in L)$ to the subsequence $X : (x_j : j \in J)$ we get a subsequence of length $|J| + h = h + r$ having at least $(r+1)$ h-sums with weights w'_i. So, adding a weighted sum of the remaining $k + r - (r + h) = k - h$ elements of the sequence S with the remaining (unused) $k - h$ weights to each of the above $(r+1)$ h-sums, we get at least $(r+1)$ k-sums with given weights w_i's.

This completes the proof. $\square$.

Chapter 3

Weighted Subsequence Sums I

3.1 Introduction.

In this chapter, we begin by defining two combinatorial group invariants, namely the *Davenport* constant and the *EGZ* constant, for a finite abelian group G and their generalizations. We mention few of the important results regarding this constants, including a result of Gao [34], establishing a link between them. After defining subsequence sums in Section 3.3, we shall present our work from [26] in Sections 3.5 and 3.6.

3.2 Generalizations of Two Combinatorial Group Invariants

Let G be an abelian group of order n, written additively. The *Davenport* constant of G, denoted by $D(G)$, is defined to be the smallest natural number t such that any sequence of elements of G of length t has a non-empty subsequence whose sum

is zero (the identity element of the group).
Another interesting constant $E(G)$, (as defined in Remark 1.3) is defined to be the smallest natural number t such that any sequence of elements of G of length t has a subsequence of length n whose sum is zero. In the case G is cyclic, we simply denote $D(G)$ (resp. $E(G)$) by $D(n)$ (resp. $E(n)$).
By the discussion at the beginning of Chapter 1, we have $D(n) \leq n$. Indeed, by considering the sequence of $n-1$ 1's, equality follows, i.e. $D(n) = n$. From the EGZ theorem (see Remark 1.2 of Chapter 1), we know that $E(n) = 2n - 1$.

Remark 3.1. As was observed in some of the early papers on the subject (see [58], for instance), the problem of finding $D(G)$ has been proposed by H. Davenport [28], in the connection with the fact that if G is the class group of an algebraic number filed $\mathbb{K}$, then $D(G)$ is maximal number of prime ideals (with multiplicity) necessary in the decomposition of an arbitrary irreducible integer in $\mathbb{K}$. However, it should be noticed that K. Rogers [63] had studied this constant in connection with the same question (see also [57]).

One of the most important applications of *Davenport* constant can be seen in the proof of the infinitude of Carmichael numbers by Alford, Granville and Pomerance [15]. It was also useful in another paper on Number Theory written by Brüdern and Godinho [21].
Determining the exact value of above constants for general abelian groups is an important and difficult question; current state of knowledge is rather limited.

Exact values for $D(G)$ were obtained by J. E. Olson when G is either a p-group ([58]) or a finite abelian group of rank 2 ([59]).
The constants $D(G)$ and $E(G)$ were being studied independently until Gao [34] (see also [41], Proposition 5.7.9) established the following result connecting these two invariants.

$$E(G) = D(G) + n - 1. \tag{3.1}$$

It is conjectured in [40], that the relation (3.1) holds for any finite non-abelian group G, where the authors proved it for the dihedral groups of order $2p$, with large exponent ($p \geq 4001$). Further, the conjecture is proved for several classes of non-abelian groups in [18].

It should be noted that Gao and Thangadurai [39] extended the above result of Gao [34] to non-abelian groups, with the condition on the repetition of an element.

Generalizations of the constants $E(G)$ and $D(G)$ with weights were considered in [6] and [12] for finite cyclic groups and later in [5], generalizations for an arbitrary finite abelian group G were introduced.

Given an abelian group G of order n, and a finite non-empty subset A of integers, the *Davenport* constant of G with weights A, denoted by $D_A(G)$, is defined to be the least positive integer t such that for every sequence $(x_1, \cdots, x_t)$ with $x_i \in G$, there exists a non-empty subsequence $(x_{j_1}, \cdots, x_{j_l})$ and $a_i \in A$ such that $\sum_{i=1}^{l} a_i x_{j_i} = 0$. Similarly, $E_A(G)$ is defined to be the least positive integer t such that every sequence of elements of G of length t has a subsequence $(x_{j_1}, \cdots, x_{j_n})$ (of length $n = |G|$) such that $\sum_{i=1}^{n} a_i x_{j_i} = 0$, for some $a_i \in A$.

In several papers ([6], [53], [42], [9], [4], [68]) the problem of determining the exact values of $E_A(\mathbb{Z}/n\mathbb{Z})$ and $D_A(\mathbb{Z}/n\mathbb{Z})$ has been taken up for various weight sets A. For a detailed exposition on this theme, we refer to the expository article [3].

For the case $A = \{1, -1\}$, authors in [6] showed that

$$E_A(n) = n + \lfloor \log_2 n \rfloor.$$

In the same paper [6], it was conjectured that for $A = (\mathbb{Z}/n\mathbb{Z})^*$ (the group of units modulo n), $E_A(n) = n + \Omega(n)$. As has been mentioned in the abstract, $\Omega(n)$ denotes the number of prime factors of n, counted with multiplicities. This conjecture was settled independently by F. Luca [53] and S. Griffiths [42]. In [26] we obtained two results which are related to this. We present these results in the subsequent sections.

3.3 Subsequence Sums

Let $m = n+r$ with $r \geq 1$. Given a sequence $x_1, x_2, \ldots, x_m$ in $\mathbb{Z}/n\mathbb{Z}$ and a subset $A \subseteq \mathbb{Z}/n\mathbb{Z}$, $A \neq \emptyset$, an A-weighted n-sum is a sum of the form $a_1 x_{i_1} + a_2 x_{i_2} + \cdots + a_n x_{i_n}$, where $i_1, i_2, \ldots, i_n$ are distinct indices and $a_i \in A$. In the case $A = \{1\}$, we simply write an n-sum instead of A-weighted n-sum.

Suppose we are given a sequence S of m elements (not necessarily distinct) of the group $\mathbb{Z}/n\mathbb{Z}$. Clearly, by definition of $E_A(n)$, if $m \geq E_A(n)$ then S has an A-weighted n-sum (subsequence sum) which is 0. So, it is natural to ask the following question :

Question : If S does not have an A-weighted n-sum which equals 0, then what can be said about the total number of A-weighted n-sums of the sequence S ?

We took up this question in our work [26], with $A = (\mathbb{Z}/n\mathbb{Z})^*$. We derived a lower bound on the number of A-weighted n-sums of such a sequence (which necessarily has length at most $E_A(n) - 1$). This is related to above mentioned results of F. Luca [53] and S. Griffiths [42].

Before proceeding, we have the following remarks.

Remark 3.2. Using definitions we have the following inequality,

$$|G| + D_A(G) - 1 \leq E_A(G) \leq 2|G| - 1. \tag{3.2}$$

For, the upper bound follows from Theorem 1.4 and the fact that $A \neq \phi$. By the definition of $D_A(G)$, there is a sequence S_1 of elements of G with length $D_A(G)-1$, which do not have an A-weighted sum (of any length ≥ 1) which is 0. By considering the sequence $S \; : \; S_1, \underbrace{0, 0, \cdots, 0}_{|G|-1 \text{ times}}$ of length $|G| + D(G) - 2$, we obtain the first inequality.

Remark 3.3. Let $A = \{1\}$ and a sequence $S \; : \; (a_1, a_2, \cdots, a_{n+r})$ of integers. It follows from Theorem 1.3 of Bollobás and Leader that if S has no A-weighted n-sum equaling 0, the number of A-weighted n-sums is at least $r+1$. This result is best possible, as can be seen from the following sequence, where the group $G = \mathbb{Z}/n\mathbb{Z}$:

$$\underbrace{0, 0, \cdots, 0}_{\text{n}-1 \text{ times}}, \underbrace{1, 1, \cdots, 1}_{\text{r}+1 \text{ times}}.$$

As mentioned at the end of Remark 1.3 (of Chapter 1), Y. O. Hamidoune [48] obtained the following generalization of Theorem 1.3, by studying subsequence sums of different length (not necessarily of length $|G|$).

Theorem 3.1 (Hamidoune) *Let $S \; : \; (a_1, a_2, \cdots, a_m)$ be a sequence of elements from a finite abelian group G. Let $l \in \mathbb{N}$ and assume $1 \leq l \leq m \leq 2l-1$. Then, one of the following holds :*

(i) The number of l-sums of S is at least $m - l + 1$.

(ii) There exists an $i \in \{1, 2, \cdots, m\}$, such that the element la_i is an l-sum.

Note that by taking $l = |G|$ and $m = |G| + r$ in above theorem, we deduce Theorem 1.3 at once.

Now we state our two results from [26]. Recall that, for a positive integer n, the notation $[n]$ is used for the set $\{1, 2, \cdots, n\}$. Also, $\varphi(n)$ is the number of integers m,

$1 \le m \le n$ which are co-prime to n. Further, by $\Omega(n)$ (resp. $\omega(n)$) we denote the number of prime factors of n counted with multiplicity (resp. without multiplicity).

Theorem 3.2 *Let p be any prime, $\alpha \ge r \ge 1$ and $A = (\mathbb{Z}/p^\alpha\mathbb{Z})^*$, the set of all units modulo p^α. Given a sequence $X = \{x_i\}_{i=1}^{p^\alpha+r}$ of integers, let*

$$S = \left\{ \sum_{i \in I} w_i x_i \pmod{p^\alpha} : I \subseteq [p^\alpha + r] \text{ with } |I| = p^\alpha,\ w_i \in A \right\}.$$

If $0 \notin S$, then $|S| \ge p^{r+1} - p^r$.

Theorem 3.3 *Let n be an odd integer, $r \ge 1$ and $A = (\mathbb{Z}/n\mathbb{Z})^*$. Given a sequence $X = \{x_i\}_{i=1}^{n+r}$ of integers, let*

$$S = \left\{ \sum_{i \in I} w_i x_i \pmod{n} : I \subseteq [n + r] \text{ with } |I| = n,\ w_i \in A \right\}$$

and $0 \notin S$. Then, there exist primes $p_1, p_2, \ldots, p_{r+1}$ such that

$$|S| \ge \varphi(p_1)\varphi(p_2)\cdots\varphi(p_{r+1}) \text{ with } p_1 p_2 \cdots p_{r+1} | n.$$

Proofs of the above results will be given in the subsequent sections. We make few remarks before proceeding.

Remark 3.4. As will be observed in Remarks 3.6 and 3.7, when n is a positive odd integer or a prime power, the results of Luca [53] and Griffiths [42] follows from Theorems 3.2 and 3.3. More precisely, given a sequence of integers $X = \{x_1, x_2, \ldots, x_{n+\Omega(n)}\}$, we have $0 \in S$. However, it should be noted that the most difficult part of the results of Luca and Griffiths, is the case when n is even; it will be interesting to extend our results to cover this case as well.

Remark 3.5. Let A be as stated in either of the theorems above. It was shown in [6], by considering the sequence,

$$1,\ p_1,\ p_1p_2, \cdots,\ p_1p_2 \ldots p_{\Omega(n)-1},$$

where $n = p_1 p_2 \dots p_{\Omega(n)}$ is the prime factorization of n, that $D_A(n) \geq \Omega(n) + 1$, for any n.

In particular, when n is a prime power or an odd integer, from above theorems we get $E_A(n) \leq n + \Omega(n)$. Hence using lower bound from equation (3.2), we get that

$$\begin{aligned} n + \Omega(n) &\leq n + D_A(n) - 1 \\ &\leq E_A(n) \\ &\leq n + \Omega(n). \end{aligned}$$

Therefore, we have

$$E_A(n) = n + D_A(n) - 1 = n + \Omega(n), \tag{3.3}$$

$$\text{and hence } D_A(n) = \Omega(n) + 1. \tag{3.4}$$

Thus the most important part of our Theorems 3.2 and 3.3 is in the situation where the sequence X has length t with $n + 1 \leq t < n + \Omega(n)$ and the condition $0 \notin S$ implies a lower bound of $|S|$.

3.4 Relation between $E_A(G)$ and $D_A(G)$

The results in the special cases observed in several papers, [6], [53], [42], [12], [68], lead (see [5], [12]) to the expectation that a weighted form of Gao's result (3.1) may holds. That is for any subset A of $\{1, 2, \cdots n\}$, $A \neq \emptyset$ and for a finite abelian group G of order n, following relation was conjectured :

$$E_A(G) = |G| + D_A(G) - 1 \tag{3.5}$$

Much of the work of [5], [12], [9], [72], [68] and [69] was motivated by or devoted to establishing the relation (3.5). When $n = p$, a prime, the method employed in [12] while determining the value of $E_A(p)$ in the case $A = \{1, 2, \cdots, r\}$, for $1 < r < p$, gives the relation (3.5), for any non-empty subset $A \subset \{1, 2, \cdots, p-1\}$ and $G = \mathbb{Z}/p\mathbb{Z}$. Another proof of this case was recently given in [11] by using the theory of permanents. Of particular note is the very recent result of Yuan and Zeng [72], establishing the relation (3.5) for the cyclic case.

For a general finite abelian group G, the following conditional result was proved by Adhikari and Chen [5].

Theorem 3.4 (Adhikari and Chen) *Let G be a finite abelian group of order n, and $A = \{a_1, a_2, \cdots, a_r\}$ be a finite subset of $\mathbb{Z}$ with $|A| = r \geq 2$ and*

$$\gcd(a_2 - a_1, a_3 - a_1, \cdots, a_r - a_1, n) = 1.$$

Then, we have $E_A(G) = n + D_A(G) - 1$.

We note that the above result does not include the result (3.1) of Gao which corresponds to the case $|A| = 1$.

Recently, Grynkiewicz, Marchan and Ordaz [45] proved the result (3.5) unconditionally for an arbitrary finite abelian group G.

3.5 Some Lemmas

For the proof of Theorems 3.2 and 3.3 we will need two lemmas.
The following lemma from [53] will be used often.

Lemma 3.1 *Let $X = \{x_1, x_2, \ldots, x_m\}$ be a sequence of integers of length m. For a prime p, we write, $X_p = \{i \in [m] : x_i \not\equiv 0 \pmod p\}$.*

i) Assume that n is odd and that $|X_p| \geq 2$ for all primes $p|n$. Then for all integers a, the equation

$$\sum_{i\in[m]} w_i x_i \equiv a \pmod{n}$$

admits a solution $w_1, w_2, \ldots, w_m \in (\mathbb{Z}/n\mathbb{Z})^*$.

ii) Assume that n is even, that $|X_p| \geq 2$ for all $p|n$, and that further $|X_2|$ is even. Then, for all even integers a, the equation

$$\sum_{i\in[m]} w_i x_i \equiv a \pmod{n}$$

admits a solution $w_1, w_2, \ldots, w_m \in (\mathbb{Z}/n\mathbb{Z})^*$.

Using Lemma 3.1, we prove the following :

Lemma 3.2 *Let p be any prime. Given a sequence $\{x_i\}_{i=1}^{p^\alpha+r}$, $\alpha \geq r \geq 1$ of integers such that $p^{\alpha-r}|x_i$, for each i, there exists $I \subseteq [p^\alpha + r]$, $|I| = p^\alpha$ such that $\sum_{i\in I} w_i x_i \equiv 0 \pmod{p^\alpha}$, for some $w_i \in (\mathbb{Z}/p^\alpha\mathbb{Z})^*$.*

Proof of Lemma 3.2. We distinguish two cases.

Case 1. p is an odd prime.

We proceed by induction on r. Take $r = 1$. If there are two indices i and j ($i \neq j$) such that

$$\frac{x_i}{p^{\alpha-1}},\ \frac{x_j}{p^{\alpha-1}} \not\equiv 0 \pmod{p}, \tag{3.6}$$

then without loss of generality let us assume that $i = 1,\ j = 2$. By Lemma 3.1, there exist $a_1,\ a_2 \in (\mathbb{Z}/p^\alpha\mathbb{Z})^*$ such that

$$a_1\frac{x_1}{p^{\alpha-1}} + a_2\frac{x_2}{p^{\alpha-1}} \equiv \sum_{i=3}^{p^\alpha} \frac{x_i}{p^{\alpha-1}} \pmod{p}.$$

Hence $a_1x_1 + a_2x_2 - x_3 - \cdots - x_{p^\alpha} \equiv 0 \pmod{p^\alpha}$.

On the other hand, if there are no two indices i and j such that (3.6) holds, then without loss of generality,

$$\frac{x_1}{p^{\alpha-1}} \equiv \frac{x_2}{p^{\alpha-1}} \equiv \cdots \equiv \frac{x_{p^\alpha}}{p^{\alpha-1}} \equiv 0 \pmod{p},$$

and hence $x_1 + x_2 + \cdots + x_{p^\alpha} \equiv 0 \pmod{p^\alpha}$ and we are through.

Now, let $r \geq 2$ and assume the induction hypothesis. If there exist $i \neq j$ such that

$$\frac{x_i}{p^{\alpha-r}},\ \frac{x_j}{p^{\alpha-r}} \not\equiv 0 \pmod{p}, \tag{3.7}$$

then as before assuming $i = 1,\ j = 2$, by Lemma 3.1, there exist $a_1,\ a_2 \in (\mathbb{Z}/p^\alpha\mathbb{Z})^*$ such that

$$a_1\frac{x_1}{p^{\alpha-r}} + a_2\frac{x_2}{p^{\alpha-r}} \equiv \sum_{i=3}^{p^\alpha} \frac{x_i}{p^{\alpha-r}} \pmod{p^r}.$$

Hence $a_1x_1 + a_2x_2 - x_3 - \cdots - x_{p^\alpha} \equiv 0 \pmod{p^\alpha}$.

As before, if there are no two indices i and j such that (3.7) holds, then without loss of generality,

$$\frac{x_1}{p^{\alpha-r}} \equiv \frac{x_2}{p^{\alpha-r}} \equiv \cdots \equiv \frac{x_{p^\alpha+r-1}}{p^{\alpha-r}} \equiv 0 \pmod{p},$$

and hence $p^{\alpha-(r-1)}|x_i$ for each $i \in [p^\alpha + (r-1)]$ and we are through by the induction hypothesis.

Case 2. $p = 2$.

We still proceed by induction on r. Take $r = 1$. If there are two indices i and j such that

$$\frac{x_i}{2^{\alpha-1}} \not\equiv \frac{x_j}{2^{\alpha-1}} \pmod{2}, \tag{3.8}$$

then without loss of generality let us assume that $i = 1,\ j = 2$ and further

$$\frac{x_1}{2^{\alpha-1}} \equiv 0 \pmod 2,\ \frac{x_2}{2^{\alpha-1}} \equiv 1 \pmod 2.$$

Now,

$$\sum_{i=3}^{2^\alpha+1} \frac{x_i}{2^{\alpha-1}} \equiv 0 \pmod 2 \Longrightarrow x_1 + x_3 + x_4 \cdots + x_{2^\alpha+1} \equiv 0 \pmod{2^\alpha}$$

and

$$\sum_{i=3}^{2^\alpha+1} \frac{x_i}{2^{\alpha-1}} \equiv 1 \pmod 2 \Longrightarrow x_2 + x_3 + x_4 \cdots + x_{2^\alpha+1} \equiv 0 \pmod{2^\alpha},$$

and we are through.

If there are no two indices i and j such that (3.8) holds, then

$$\frac{x_1}{2^{\alpha-1}} \equiv \frac{x_2}{2^{\alpha-1}} \equiv \cdots \equiv \frac{x_{2^\alpha+1}}{2^{\alpha-1}} \pmod 2,$$

and hence $x_1 + x_2 + \cdots + x_{2^\alpha} \equiv 0 \pmod{2^\alpha}$.

Now, let $r \geq 2$ and we assume the induction hypothesis.

If there are not more than one $i \in [p^\alpha + r]$ such that $x_i/2^{\alpha-r} \equiv 1 \pmod 2$, then without loss of generality let us assume that

$$\frac{x_1}{2^{\alpha-r}} \equiv \frac{x_2}{2^{\alpha-r}} \equiv \cdots \equiv \frac{x_{2^\alpha+r-1}}{2^{\alpha-r}} \equiv 0 \pmod 2,$$

which would imply that $2^{\alpha-(r-1)}|x_i$, for each $i \in [2^\alpha + (r-1)]$ and we are through by the induction hypothesis.

If there are more than one $i \in [p^\alpha + r]$ such that $x_i/2^{\alpha-r} \equiv 1 \pmod 2$, then it is easy to observe that we can choose $I \subseteq [2^\alpha + r]$, $|I| = 2^\alpha$ such that

$$\left|\left\{ i \in I : \frac{x_i}{2^{\alpha-r}} \equiv 1 \pmod 2 \right\}\right|$$

is even and at least two, and therefore by part (ii) of Lemma 1, for some $a_i \in (\mathbb{Z}/2^r\mathbb{Z})^*$, we have

$$\sum_{i\in I} a_i \frac{x_i}{2^{\alpha-r}} \equiv 0 \pmod{2^r},$$

and hence

$$\sum_{i\in I} a_i x_i \equiv 0 \pmod{2^\alpha},$$

where $a_i \in (\mathbb{Z}/2^\alpha\mathbb{Z})^*$.

This proves the lemma. □

3.6 Proofs of Theorems 3.2 and 3.3

Proof of Theorem 3.2. Assume that $0 \notin S$. Let α_i be integers such that $p^{\alpha_i}||x_i$, (i.e. p^{α_i} is the largest power of p, that divides x_i) for $i \in [p^\alpha + r]$ and without loss of generality we can assume that

$$\alpha_1 = \min\{\alpha_i : i \in [p^\alpha + r]\}.$$

If $\alpha_1 \geq \alpha - r$ then we have $p^{\alpha-r}|x_i$ for each i and therefore Lemma 3.2 contradicts the assumption that $0 \notin S$.

Therefore we must have $\alpha_1 \leq \alpha - (r+1)$. Consider the following subset of S,

$$S' = \{ax_1 + x_2 + \cdots + x_{p^\alpha} : a \in (\mathbb{Z}/p^{\alpha-\alpha_1}\mathbb{Z})^*\}.$$

Observe that, for $a, b \in (\mathbb{Z}/p^{\alpha-\alpha_1}\mathbb{Z})^*$,

$$\begin{aligned} ax_1 \equiv bx_1 \pmod{p^\alpha} \quad &\Rightarrow \quad a\frac{x_1}{p^{\alpha_1}} \equiv b\frac{x_1}{p^{\alpha_1}} \pmod{p^{\alpha-\alpha_1}} \\ &\Rightarrow \quad a \equiv b \pmod{p^{\alpha-\alpha_1}}. \end{aligned}$$

Thus,

$$\begin{aligned} |S| &\geq |S'| \\ &\geq \varphi(p^{\alpha-\alpha_1}) \geq \varphi(p^{r+1}) \\ &= p^{r+1} - p^r. \end{aligned}$$

Hence the theorem. □.

Remark 3.6. We observe that if the length of X is $p^\alpha + \alpha - 1 \geq p^\alpha + 1$, then $0 \notin S$ would imply that $|S|$ is large; more precisely, we have $|S| \geq \varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$.

Taking $r = \alpha$, in Lemma 3.2 (and of course, in the above theorem) the results of Luca and Griffiths follows in the particular case when n is a prime power.

Proof of Theorem 3.3. Let

$$d = \gcd(x_1, x_2, \ldots, x_{n+r}, n).$$

Write $y_i = x_i/d$ for all $i \in [n+r]$ and consider the sequence
$Y = \{y_1, \ldots, y_{n+r}\}$. We observe that

$$\gcd(y_1, y_2, \ldots, y_{n+r}, n/d) = 1. \tag{3.9}$$

Now, we proceed to prove our theorem by induction on r. Let $r = 1$.

If $|Y_p| \geq 2$ for all primes p dividing n/d, then since $2\omega(n/d) \leq n/d \leq n$, we can find a subsequence T of length n of Y such that $|T_p| \geq 2$ for all primes p dividing n/d and by Part (i) of Lemma 3.1,

$$\sum_{i \in I} w_i y_i \equiv 0 \pmod{n/d},$$

for some $I \subseteq [n+r]$ with $|I| = n,\ w_i \in A$, which is a contradiction to the fact that $0 \notin S$.

Therefore, there exists a prime p_1 dividing n/d, such that $|Y_{p_1}| \le 1$. By (3.9), we have $|Y_{p_1}| = 1$.

Without loss of generality we assume that $y_{n+r} \not\equiv 0 \pmod{p_1}$ and $y_i \equiv 0 \pmod{p_1}$ for each $i \in [n+r-1]$.

If $y_1 \equiv y_2 \equiv \cdots \equiv y_n \equiv 0 \pmod{n/d}$ then $0 \in S$. Thus, without loss of generality, there exists a prime p_2, such that $p_2^t | \frac{n}{p_1 d}$, $y_1 \not\equiv 0 \pmod{p_1 p_2^t}$ and $y_1 \equiv 0 \pmod{p_1 p_2^{t-1}}$.

Consider the following subset of S:

$$S' = \{ax_1 + bx_{n+r} + x_2 + x_3 \cdots + x_{n-1} : a \in A', b \in B'\},$$

where A' and B' are subsets of $(\mathbb{Z}/n\mathbb{Z})^*$ which are respectively the lifts of $(\mathbb{Z}/p_2\mathbb{Z})^*$ and $(\mathbb{Z}/p_1\mathbb{Z})^*$ in $(\mathbb{Z}/n\mathbb{Z})$.

Now we observe that, if $a_1x_1 + b_1x_{n+r} \equiv a_2x_1 + b_2x_{n+r} \pmod{n}$, then $(b_2 - b_1)y_{n+r} \equiv 0 \pmod{p_1}$ and hence $b_1 = b_2$. Therefore, $(a_1 - a_2)y_1 \equiv 0 \pmod{n/d}$ which implies that $(a_1 - a_2)y_1 \equiv 0 \pmod{p_1p_2^t}$ so that $(a_1 - a_2)(y_1/p_2^{t-1}p_1) \equiv 0 \pmod{p_2}$ and hence $a_1 = a_2$.

Thus,

$$\begin{aligned} |S| &\ge |S'| \\ &\ge |A'||B'| \\ &= \varphi(p_1)\varphi(p_2), \end{aligned}$$

where $p_1p_2|n$.

Now, let $r \ge 2$ and assume the induction hypothesis.

As before, there exists a prime $p_1|(n/d)$ such that $|Y_{p_1}| = 1$. Without loss of generality we can assume that $y_{n+r} \not\equiv 0 \pmod{p_1}$ and $y_i \equiv 0 \pmod{p_1}$ for each $i \in [n+r-1]$.

Consider the sequence $\{y_i/p_1\}_1^{n+r-1}$. Write $n' = n/(p_1 d)$.

We claim that for some $I \subseteq [n+r-1]$, with $|I| = n'+r-1$, we have

$$0 \notin \left\{ \sum_{i \in J} w_i y_i / p_1 \pmod{n'} : J \subseteq I, |J| = n' \text{ and } w_i \in (\mathbb{Z}/n'\mathbb{Z})^* \right\}.$$

For, otherwise we get disjoint subsets $I_1, I_2, \ldots, I_{p_1 d}$ of $[n+r-1]$, with $|I_j| = n'$ such that for each j, we have

$$\sum_{i \in I_j} w_{ij} y_i / p_1 \equiv 0 \pmod{n'},$$

for some $w_{ij} \in (\mathbb{Z}/n'\mathbb{Z})^*$. Now, by lifting each w_{ij} in A (and using the same notation w_{ij} for the lift) we get,

$$\sum_{j=1}^{p_1 d} \left(\sum_{i \in I_j} w_{ij} x_i \right) \equiv 0 \pmod{n}.$$

Thus, $0 \in S$, a contradiction. Hence the claim is established.

Without loss of generality, let $I = [n'+r-1]$. By induction hypothesis there exist primes $p_2, p_3, \ldots p_{r+1}$ such that

$$\left| \left\{ \sum_{i \in J} w_i y_i / p_1 \pmod{n'} : J \subseteq I,\ |J| = n' \text{ and } w_i \in (\mathbb{Z}/n'\mathbb{Z})^* \right\} \right|$$

is at least $\varphi(p_2)\varphi(p_3)\cdots\varphi(p_{r+1})$ and $p_2 p_3 \cdots p_{r+1} | n'$.

Therefore, by lifting each w_i in A and considering the set

$$S' := \left\{ \sum_{i \in J} w_i x_i \pmod{n} : J \subseteq I,\ |J| = n' \text{ and } w_i \in A \right\},$$

we have

$$l = |S'| \geq \varphi(p_2)\varphi(p_3)\cdots\varphi(p_{r+1}).$$

Let

$$S' = \{t_1, t_2, \ldots, t_l\}.$$

Note that, $t_i \equiv 0 \pmod{p_1}$ for each i.

Now consider the set $T' = \{ax_{n+r} + t_i : i \in [l],\ a \in B'\}$ (recall that B' is the lift of $(\mathbb{Z}/p_1\mathbb{Z})^*$ in $\mathbb{Z}/n\mathbb{Z}$). We observe that, if $ax_{n+r} + t_i \equiv bx_{n+r} + t_j \pmod{n}$, then $(a-b)y_{n+r} \equiv 0 \pmod{p_1}$ and hence $a = b$. Therefore, $t_i \equiv t_j \pmod{n}$ and hence $t_i = t_j$.

Thus $|T'| \geq |B'|l \geq \varphi(p_1)\varphi(p_2)\cdots\varphi(p_{r+1})$, with $p_1p_2\cdots p_{r+1}|n$.

Since $n + r - ((n' + r - 1) + 1) = n - n'$, by considering distinct indices $i_1, i_2, \ldots, i_{n-n'-1} \in [n+r] \setminus (I \cup \{n+r\})$ we get

$$T' + x_{i_1} + x_{i_2} + \cdots + x_{i_{n-n'-1}} \subseteq S.$$

Hence $|S| \geq \varphi(p_1)\varphi(p_2)\cdots\varphi(p_{r+1})$, with $p_1p_2\cdots p_{r+1}|n$.

This completes the proof. $\square$.

Remark 3.7. When length of X is $n + \Omega(n) - 1$, and $n = p_1 \ldots p_r$, then $0 \notin S$ would imply that $|S| \geq (p_1 - 1)\ldots(p_r - 1)$. If X is of length $n + \Omega(n)$, then a contradiction would imply that $0 \in S$. Thus, the results of Luca and Griffiths follows in the particular case when n is an odd integer.

Chapter 4

Weighted Subsequence Sums II

4.1 Introduction

We defined the generalized *Davenport* constant $D_A(G)$ and the generalized *EGZ* constant $E_A(G)$, in Section 3.2 of the previous chapter. We also asked the following question for the cyclic group $\mathbb{Z}/n\mathbb{Z}$ (see Section 3.3 of Chapter 3):

Given a sequence S of elements of G, if S does not have an A-weighted zero-sum subsequence of length $|G|$, what can be said about the number of distinct $|G|$-length A-weighted subsequence sums ?

In Chapter 3, we answered this question by giving a lower bound on the number of $|G|$-sums, in particular case when G is cyclic group of order n (a prime power or an odd integer) and the set $A = (\mathbb{Z}/n\mathbb{Z})^*$. In the present chapter we take up the problem with the set $A = \{1, -1\}$ and a general abelian group G of finite order. We present our work from [7], which in particular gave an alternate proof of the main result of [6] (for more details refer to Remark 4.4).

More precisely, we have proved the following theorem (for some terminology in the statement of the theorem, one may look into Section 4.3) :

Theorem 4.1 *Let G be a finite abelian group of order n and let it be of the form $G \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}$, where $1 < n_1 | n_2 | \cdots | n_r$. Let $A = \{1, -1\}$ and k be a natural number satisfying $k \geq 2^{r'-1} - 1 + \frac{r'}{2}$, where $r' = |\{i \in [r] : n_i \text{ is even}\}|$. Then, given a sequence $S = (x_1, x_2, \cdots, x_{n+k})$, with $x_i \in G$, if S has no A-weighted zero-sum subsequence of length n, then there are at least $2^{k+1} - \delta$ distinct A-weighted n-sums, where*

$$\delta = \begin{cases} 1 & \text{if } 2 \mid n \\ 0 & \text{otherwise.} \end{cases}$$

For a finite abelian group G of order n, Gao and Leader [38] obtained some result on the description of some sequences which do not have 0 as an n-sum and at which the minimum number of n-sums is attained.

4.2 Relating sums to $|G|$-sums

Given an abelian group G of order k and a sequence S of elements of G. Family of k-sums of a sequence S has been studied by several authors. We give a brief account of results related to sums and k-sums of a sequence of elements of G.

As can be seen from the result (3.1) of Gao [34], the study of k-sums is closely related to the study of subsequence sums. J. E. Olson [61] gave a sufficient condition for the family of k-sums from a sequence S of length $2k-1$ to be the entire group G. This result was extended by W. Gao [35] to deal with sequences of general length.

Hamidoune, Ordaz and Ortuño [49] gave a sufficient condition for 0 to be an k-sum from a sequence $S : a_1, a_2, \cdots, a_r$, in terms of the number of a_i that are allowed to assume the same value.

Bollobás and Leader [20] conjectured the following extension of their result (Theorem 1.3) and the result of Gao [34] (the relation (3.1)).

Conjecture 4.1. The minimum number of k-sums for a sequence $S : a_1, \cdots, a_{k+r}$ of elements from G that does not have 0 as a k-sum is attained at the sequence

$$b_1, b_2, \cdots, b_{r+1}, \underbrace{0, 0, \cdots, 0}_{\text{k}-1 \text{ times}}$$

where $b_1, b_2, \cdots, b_{r+1}$ is chosen to minimize the number of sums (of length ≥ 1) without 0 being a sum.

Remark 4.1. The problem of minimizing the number of sums without 0 being a sum, can easily be seen solved as follows :

If $b_1, b_2, \cdots, b_{r+1}$ is a sequence that does not have 0 as a sum (of any length ≥ 1) then it has at least $r+1$ distinct sums :

$$b_1, b_1 + b_2, \cdots, b_1 + b_2 + \cdots + b_{r+1}.$$

Moreover, this is best possible as can be seen by considering the sequence

$$\underbrace{1, 1, \cdots, 1}_{\text{r+1 times}},$$

of elements from $\mathbb{Z}/k\mathbb{Z}$ (here $r + 1 \leq k - 1$).

Gao and Leader confirmed the above conjecture in [38].

Bialostocki and Dierker [19] generalized the EGZ theorem by studying sequences of length $2k-2$ and characterizing those without zero-sum subsequences of length k. They proved that such a sequence can only occur when G is cyclic and the sequence contains exactly two group elements, each occurring exactly $k-1$ times. Exploiting a distinctness assumption, Grynkiewicz [44] obtained a result that generalizes the above mentioned result of Bialostocki and Dierker [19], and also solved a conjecture of Hamidoune [48], and extended some results of Hamidoune [48], and of Hamidoune, Ordaz and Ortuño [49].

4.3 Notations and Preliminaries

Let G be a finite abelian group of order n, written additively and let A be a non-empty subset of $\{1, 2, \cdots, n-1\}$. Given a sequence $S : (s_1, s_2, \cdots, s_r)$ of elements of G and $\bar{a} = (a_1, a_2, \cdots, a_r) \in A^r$, we define $\sigma(S) = \sum_{i=1}^{r} s_i$ and $\sigma^{\bar{a}}(S) = \sum_{i=1}^{r} a_i s_i$. If $\sigma(S) = 0$ (resp. $\sigma^{\bar{a}}(S) = 0$ for some $\bar{a} \in A^r$), we say that S is a zero-sum (resp. an A-weighted zero-sum) sequence.

For $x \in G$, Ax will denote the following subset of the group G,

$$Ax := \{ax : a \in A\}.$$

If H is a subgroup of G, then $\phi_H : G \to G/H$ will denote the natural homomorphism and given a sequence $S : (s_1, s_2, \cdots, s_r)$ of elements of G, $\phi_H(S)$ will denote the sequence $(\phi_H(s_1), \phi_H(s_2), \cdots, \phi_H(s_r))$ with elements in G/H.

The length of a sequence S will be denoted by $|S|$; we think that this will not have any confusion with the usual notation $|G|$ used to denote the order of a finite group G.

For a subsequence S' of a sequence S, we use $S \setminus S'$ to denote the sequence obtained by removing the elements of the subsequence S' from S.

Generalizing a definition in [71], we call a sequence S with elements in G an *A-weighted zero-smooth sequence* if for any $1 \le l \le |S|$, there exists an A-weighted zero-smooth subsequence of S with length l. When $A = \{1\}$, S is simply called a zero-smooth sequence.

Remark 4.2. If S is a zero-smooth sequence then it is also an A-weighted zero-smooth sequence. This is obvious, as $A \neq \emptyset$.

Remark 4.3. We observe that if $U = (u_1, u_2, \cdots, u_r)$ and $V = (v_1, v_2, \cdots, v_s)$ are sequences of elements of G such that U is an A-weighted zero-smooth sequence and V is an A-weighted zero-sum sequence with $|V| \leq |U| + 1$, then the sequence $(u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_s)$, obtained by appending V to U, is an A-weighted zero-smooth sequence.

We shall need the following result of Yuan and Zeng [71] on the existence of zero-smooth subsequence.

Theorem 4.2 (Yuan, Zeng) *Let G be an abelian group of order n. Let S be a sequence with elements in G such that $|S| \geq n + D(G) - 1$ and the element 0 is repeated maximum number of times in S. Then there exists a subsequence S_1 of S which is zero-smooth and $|S_1| \geq |S| - D(G) + 1$.*

M. Kneser [52] (one may also see [56] or [67]) generalized the Cauchy-Davenport Theorem (Theorem 1.2) as follows :

Theorem 4.3 (Kneser) *Let G be an abelian group, and let A and B be finite, non-empty subsets of G. Let $H = Stab(A + B)$. Then*

$$|A + B| \geq |A + H| + |B + H| - |H|.$$

We also need a very useful generalization of the above theorem due to DeVos, Goddyn and Mohar [29].

Let $\mathcal{A} = (A_1, A_2, \cdots, A_r)$, where $r \geq n = |G|$, be a sequence of finite non-empty subsets of G. Let $\sum_n(\mathcal{A})$ denotes the set of all group elements representable as a sum of n elements from distinct terms of $\mathcal{A}$. i.e. we have

$$\sum_n(\mathcal{A}) = \{a_{i_1} + a_{i_2} + \cdots + a_{i_n} : 1 \leq i_1 < i_2 < \cdots < i_n \leq r\}.$$

Further, let $H = Stab(\sum_n(\mathcal{A})) := \{g \in G : g + \sum_n(\mathcal{A}) = \sum_n(\mathcal{A})\}$.

The theorem of DeVos, Goddyn and Mohar [29] is the following.

Theorem 4.4 (DeVos, Goddyn and Mohar) *With the notations as above, we have*

$$\left|\sum_n(\mathcal{A})\right| \geq |H|\left(1-n+\sum_{g\in G/H}\min\{n,|\{j : g\cap A_j \neq \emptyset\}|\}\right).$$

4.4 Proof of Theorem 4.1

In the case $r'=0$, it is possible to have $k=0$. We observe that in this case, $|S|=n$ and if $\sigma(S)=t\neq 0$, then $-\sigma(S)=-t\neq 0$. Again, n being odd, G does not have any element of order 2 and thus there are at least two distinct A-weighted n-sums, viz. t and $-t$. So, the result is true in this case and we may assume that $k\geq 1$.

If possible, we suppose that the result is not true and choose a counter example (G,S,k) with $|G|=n$ minimal.

Considering the sequence $\mathcal{A}=(A_1,A_2,\cdots,A_{n+k})$, where $A_i=Ax_i$ for each i, $1\leq i\leq n+k$, we have,

$$0 \notin \sum_n(\mathcal{A}), \tag{4.1}$$

$$\text{and} \quad \left|\sum_n(\mathcal{A})\right| < 2^{k+1}-\delta. \tag{4.2}$$

Let $L=Stab(\sum_n(\mathcal{A}))$. We claim that $L=\langle 0\rangle$.

If possible, let $L\neq\langle 0\rangle$, so that $|G/L|<n$. Writting the identity element of G/L as $\mathbf{0}$, if for every subsequence $S'=\{x_{i_1},x_{i_2},\cdots,x_{i_d}\}$ of S of length $d=|G/L|+k$, $\mathbf{0}$ is representable as a sum of $|G/L|$ elements from distinct terms of

the sequence $(\phi_L(Ax_{i_1}), \phi_L(Ax_{i_2}), \cdots, \phi_L(Ax_{i_d}))$, then we get pairwise disjoint subsequences $S_1, S_2, \cdots, S_{|L|}$, each of length $|G/L|$ and $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{|L|} \in A^{|G/L|}$ such that

$$\sigma^{\bar{a}_i}(\phi_L(S_i)) = \mathbf{0},$$

for each i, $1 \leq i \leq |L|$.

Therefore, we have

$$\begin{aligned}
\sum_{i=1}^{|L|} \sigma^{\bar{a}_i}(\phi_L(S_i)) &= \mathbf{0} \\
\Rightarrow \quad \phi_L\left(\sum_{i=1}^{|L|} \sigma^{\bar{a}_i}(S_i)\right) &= \mathbf{0}.
\end{aligned}$$

Writting $\theta = \sigma^{\bar{a}_1}(S_1) + \sigma^{\bar{a}_2}(S_2) + \cdots + \sigma^{\bar{a}_{|L|}}(S_{|L|})$, as $\theta \in L = Stab(\sum_n(\mathcal{A}))$ we also have $-\theta \in L$. Since $\theta \in \sum_n(\mathcal{A})$, we get $0 = -\theta + \theta \in \sum_n(\mathcal{A})$, contradicting to (4.1).

Hence there exists a subsequence S' of S with length $|G/L| + k$ (observe that a permissible value of k for G is obviously a permissible for G/L) such that $\mathbf{0} \notin \sum_{|G/L|}(\phi_L(\mathcal{A}'))$, where $\mathcal{A}'$ is the subsequence of $\mathcal{A}$ corresponding to the sequence S'. Hence by minimality of $|G|$, letting

$$\delta' = \begin{cases} 1 & \text{if } 2||G/L|, \\ 0 & \text{otherwise} \end{cases}$$

we have,

$$\begin{aligned}
\left|\sum_{|G/L|}(\phi_L(\mathcal{A}'))\right| &\geq 2^{k+1} - \delta' \\
&\geq 2^{k+1} - \delta.
\end{aligned}$$

And hence,

$$\left|\sum_{|G/L|}(\mathcal{A}')\right| \geq 2^{k+1} - \delta.$$

Since the length of the subsequence $\mathcal{A} \setminus \mathcal{A}'$ is $n + k - (|G/L| + k) = n - |G/L|$, we have

$$\begin{aligned}\left|\sum_{n}(\mathcal{A})\right| &\geq \left|\sum_{|G/L|}(\mathcal{A}')\right| \\ &\geq 2^{k+1} - \delta,\end{aligned}$$

a contradiction to (4.2).

Therefore we have the claim, $L = \langle 0 \rangle$. Hence by Theorem 4.4 of DeVos Goddyn and Mohar, we have

$$\left|\sum_{n}(\mathcal{A})\right| \geq 1 - n + \sum_{x\in G}\min\{n, |\{i \;:\; 1 \leq i \leq n+k, x \in A_i\}|\}.$$

Since (4.1) implies, in particular that no element of G can be in n distinct A_i's, we have

$$\begin{aligned}\left|\sum_{n}(\mathcal{A})\right| &\geq 1 - n + \sum_{x\in G}\min\{n, |\{i \;:\; 1 \leq i \leq n+k, x \in A_i\}|\} \\ &= 1 - n + \sum_{x\in G}|\{i \;:\; 1 \leq i \leq n+k, x \in A_i\}| \\ &= 1 - n + \sum_{i=1}^{n+k}|A_i|.\end{aligned}$$

Writting $t = |\{j \;:\; 1 \leq j \leq n+k,\; |A_j| = 1\}|$, from (4.1) and the above inequality we have,

$$n - 1 \geq \left|\sum_{n}(\mathcal{A})\right| \geq 1 - n + 2(n + k - t) + t,$$

and hence

$$t \geq 2(k+1).$$

Rearranging, if needed, we assume that $(x_1, x_2, \cdots, x_t)$ is the subsequence of S such that $|A_i| = |Ax_i| = 1$ for each $i,\ 1 \leq i \leq t$ and the element x_1 is repeated maximum number of times in the subsequence $(x_1, x_2, \cdots, x_t)$.

We observe that all the x_i's appearing in $(x_1, x_2, \cdots, x_t)$ are either equal to the zero element of the group or those of order 2, when n is even.

Consider the sequence $S' = (y_1, y_2, \cdots, y_{n+k})$, where $y_i = x_i - x_1$, for each $i,\ 1 \leq i \leq n+k$. Write $\mathcal{B} = (B_1, B_2, \cdots, B_{n+k})$, where $B_i = Ay_i = A(x_i - x_1)$, for each i, $1 \leq i \leq n+k$.

Observing that $|Ax_1| = 1$, if we consider a typical element

$$\epsilon_{i_1} y_{i_1} + \epsilon_{i_2} y_{i_2} + \cdots + \epsilon_{i_n} y_{i_n},$$

of $\sum_n(\mathcal{B})$, where $\epsilon_j \in \{1, -1\}$, then it can be written as:

$$\begin{aligned}\epsilon_{i_1}(x_{i_1} - x_1) + \epsilon_{i_2}(x_{i_2} - x_1) + \cdots + \epsilon_{i_n}(x_{i_n} - x_1)\\ = \epsilon_{i_1} x_{i_1} + \epsilon_{i_2} x_{i_2} + \cdots + \epsilon_{i_n} x_{i_n},\end{aligned}$$

as $\sum_{j=1}^{n} \epsilon_{i_j} x_1 = n x_1 = 0$.

Hence, $\sum_n(\mathcal{A}) = \sum_n(\mathcal{B})$ and from (4.1) and (4.2), we have

$$0 \notin \sum_n(\mathcal{B}) \tag{4.3}$$

$$\text{and} \quad \left|\sum_n(\mathcal{B})\right| < 2^{k+1} - \delta. \tag{4.4}$$

By our construction, in the subsequence $S_1 = (y_1, y_2, \cdots, y_t)$ of S', all the elements $y_i,\ 1 \leq i \leq t$, satisfy $2y_i = 0$ and the element $y_1 = 0$ is repeated maximum number of times in S_1.

Depending on the parity of n, we consider the following two cases :

Case I (n is odd). We observe that in this case, $y_i = 0$ for all $i,\ 1 \leq i \leq t$. Now, we choose a maximal A-weighted zero-sum subsequence S_2 of $S' \setminus S_1$, (we use $S' \setminus S_1$ to denote the sequence obtained by removing the elements of the subsequence S_1 from S'), possibly empty. If $|(S' \setminus S_1) \setminus S_2| \leq k$, then

$$(n+k) - |S_1| - |S_2| \leq k \quad \Rightarrow \quad n - |S_2| \leq |S_1|.$$

And hence by appending a subsequence of (zeros) S_1 of length $n - |S_2|$ to S_2 we get an A-weighted zero-sum subsequence of S' of length n, which is a contradiction to (4.3).

Thus there exists a subsequence $S_3 = (y_{j_1}, y_{j_2}, \cdots, y_{j_{k+1}})$ of $(S' \setminus S_1) \setminus S_2$ which does not have any non-empty A-weighted zero-sum subsequence, by maximality of S_2.

Consider the set,

$$X = \left\{ \sum_{i=1}^{k+1} \epsilon_i y_{j_i} : \epsilon_i \in A = \{1, -1\} \right\}.$$

If for $\epsilon_i,\ \epsilon_i' \in A = \{1, -1\}$, we have

$$\sum_{i=1}^{k+1} \epsilon_i y_{j_i} = \sum_{i=1}^{k+1} \epsilon_i' y_{j_i},$$

then writing $I = \{i : \epsilon_i \neq \epsilon_i'\}$, we get,

$$\begin{aligned} 2\sum_{i \in I} \gamma_i y_{j_i} &= 0 \\ \Rightarrow \quad \sum_{i \in I} \gamma_i y_{j_i} &= 0, \text{ (since n is odd)} \end{aligned}$$

where $\gamma_i = \epsilon_i - \epsilon_i' \in A$, for each $i \in I$. This leads to a contradiction (by appending the sequence $(y_{j_i} : i \in I)$ to S_2) to the maximality of S_2, if I is non-empty.

Thus, we have $|X| \geq 2^{k+1}$ and considering the sum of a fixed subsequence of $S' \setminus S_3$ of length $n-(k+1)$, and adding that to various sums in X, we have $|\sum_n(\mathcal{B})| \geq 2^{k+1}$ which is a contradiction to (4.4).

Case II (n is even). Put $H = \langle y_1, y_2, \cdots, y_t \rangle$. As we have already observed, $2y_i = 0$, for all $i,\ 1 \leq i \leq t$. Hence H is a subgroup of $\mathbb{Z}_2^{r'}$.

Thus,

$$|H| \leq 2^{r'} \tag{4.5}$$

and by a result of Olson [58] on the Davenport constant of p-groups,

$$D(H) \leq D(\mathbb{Z}_2^{r'}) = r' + 1. \tag{4.6}$$

Since, by our assumption, $k \geq 2^{r'-1} - 1 + \frac{r'}{2}$, by (4.5) we have,

$$|S_1| = t \geq 2(k+1) \geq 2^{r'} + r' \geq |H| + D(H) - 1.$$

Also, 0 is repeated maximum number of times in S_1.

So, we can apply Theorem 4.2 of Yuan and Zeng, and it follows that S_1 has a zero-smooth subsequence T_1 such that $|T_1| \geq |S_1| - D(H) + 1$. Therefore, from the fact $|S_1| = t \geq 2(k+1)$ and (4.6) we have

$$|T_1| \geq 2k + 2 - r'.$$

Again, since $k \geq 2^{r'-1} - 1 + \frac{r'}{2}$, we have $k - r' \geq 2^{r'-1} - 1 - \frac{r'}{2} \geq -1$, and thus

$$\begin{aligned} |T_1| &\geq 2k + 2 - r' \\ &= k + 2 + k - r' \\ &\geq k + 1. \end{aligned}$$

Now, we choose a maximal A-weighted zero-smooth subsequence T of S'. We have, $|T| \geq |T_1| \geq k+1$. Further, (4.3) implies that $|T| < n$. Say $|T| = n - l$, $l \geq 1$.

Consider the subsequence $S' \setminus T = (y_{s_1}, y_{s_2}, \cdots, y_{s_{k+l}})$, and the set

$$Y = \left\{ \sum_{i\in I} y_{s_i} : I \subset \{1, 2, \cdots, k+1\},\ I \neq \emptyset \right\}.$$

Now, if for $I \neq J,\ I \neq \emptyset,\ J \neq \emptyset$, we have

$$\sum_{i\in I} y_{s_i} = \sum_{i\in J} y_{s_i},$$

then we shall have

$$\sum_{i\in I'} \delta_i y_{s_i} = 0, \quad \delta_i \in A,$$

where $I' = (I \cup J) \setminus (I \cap J)$.

Since it is clear that I' is non-empty, and that $1 \leq |I'| \leq k+1 \leq |T|$; from the observations made in Remarks 4.2 and 4.3 of Section 4.3, by appending the subsequence corresponding to I' to T, we get a contradiction to the maximality of T. Therefore we have $|Y| = 2^{k+1} - 1$. Adding $y_{s_{k+2}} + y_{s_{k+3}} + \cdots + y_{s_{k+l}}$ to each of the distinct sums in Y, we get $2^{k+1} - 1$ distinct sums :

$$y_{s_{k+2}} + y_{s_{k+3}} + \cdots + y_{s_{k+l}} + \sum_{i\in I} y_{s_i},$$

$I \subset \{1, 2, \cdots, k+1\},\ I \neq \emptyset.$

Now, for a given $I \subset \{1, 2, \cdots, k+1\},\ I \neq \emptyset$, as $n - (|I| + l - 1) \leq n - l = |T|$, we can append an $n - (|I| + l - 1)$ length A-weighted zero-sum subsequence of T to

$$y_{s_{k+2}} + y_{s_{k+3}} + \cdots + y_{s_{k+l}} + \sum_{i\in I} y_{s_i}$$

to make an A-weighted n-sum without changing the value of the sum.

Thus, $|\sum_n(\mathcal{B})| \geq 2^{k+1} - 1$, which contradicts to (4.4).

Hence the theorem. $\square$.

Remark 4.4. It is not difficult to observe that for a finite abelian group G with $G \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}$, $1 < n_1|n_2|\cdots|n_r$, satisfying $|G| > 2^{(2^{r'-1}-1+\frac{r'}{2})}$, where $r' = |\{i \in \{1,2,\cdots,r\} : 2|n_i\}|$, and $A = \{1,-1\}$, our result (Theorem 4.1) along with some counter examples like those given in [6] (see also [10]), yields

$$|G| + \sum_{i=1}^{r} \lfloor \log_2 n_i \rfloor \leq E_A(G) \leq |G| + \lfloor \log_2 |G| \rfloor. \tag{4.7}$$

This gives the exact value of $E_A(G)$ when G is cyclic (thus giving another proof of the main result in [6]) and unconditional bounds in many cases.

However, we mention that when $A = \{1,-1\}$, finding the corresponding bounds for $D_A(G)$ for a finite abelian group G and the exact value of $D_A(G)$, when G is cyclic, is not so difficult (see [6], [10]). Therefore, from the relation

$$E_A(G) = |G| + D_A(G) - 1,$$

for an abelian group G and a non-empty subset A of $[n-1]$, (which generalizes (3.1)) established recently for cyclic groups by Yuan and Zeng [72] and soon afterwards for general finite abelian groups by Grynkiewicz, Marchan and Ordaz [45], the result (4.7) follows.

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