ON THE GROWTH OF TRACE OF POWERS OF ALGEBRAIC INTEGERS

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Abstract. Let α be a non-zero algebraic integer. In this short note, we study the growth of $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)$ as $n \to \infty$. One of the interesting results that we prove is a characterisation for α to be a root of unity which is an extension of Kronecker's Theorem. Indeed, we prove that *if a non-zero algebraic integer α is a root of unity if and only if the sequence $\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)\right)_{n\geq 1}$ is bounded. Moreover, if the sequence $\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)\right)_{n\geq 1}$ is bounded, then it is periodic.* Thus, if a non-zero algebraic integer α is not a root of unity, it is clear that the sequence $\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)\right)_{n\geq 1}$ is unbounded and hence we study the growth in the next result. These results are an indirect application of the Schmidt subspace theorem.

1. INTRODUCTION

In the literature, starting from Polya [13], there are several authors who studied the integrality condition of a given algebraic number α using the integral trace of α^n. For instance, we refer to [13], [8], [6], [12] and [2]. There is a well-known Schur-Siegel-Smyth Trace problem related to the trace of totally positive algebraic integers which has a connection with sharp Hasse - Weil bound for abelian varieties over large finite fields and suggests that *there exists a constant $\rho > 1$ such that*

$$\liminf \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]}\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha) = \rho$$

where the limit infimum runs through all the totally positive algebraic integers α. Smyth [15] proved that $\rho \geq 1.7719$ and Smith [14] proved that $\rho < 1.8984$. We refer to two recent articles on these topics, say, B. Kadets [9] and A. Smith [14] for more information.

If α is a totally positive algebraic integer of degree d, then by the definition of trace, we get

$$dA \leq \mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha) \leq dB$$

where $A = \min\{\beta : \beta \text{ is a conjugate of } \alpha\}$ and $B = \max\{\beta : \beta \text{ is a conjugate of } \alpha\}$ and hence we get

$$dA^n \leq \mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n) \leq dB^n$$

for all natural numbers n. It is natural to study similar bounds for a general nonzero algebraic integer. If α is a general nonzero algebraic integer, then there is a possibility that $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^m) = 0$ for some natural number m. Therefore, first, we observe the following.

Proposition 1.1. *Let α be a nonzero algebraic integer. Then there exists infinitely many natural numbers n such that $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n) \neq 0$.*

Another interesting result that is stronger than Proposition 1.1 is as follows.

Proposition 1.2. *Let α be an algebraic number of degree $d \geq 2$. Then for any given natural number a, we have $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n) \neq 0$ for some natural number n satisfying $a \leq n \leq a + d - 1$.*

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When $d = 2$ in Proposition 1.2, we get the following corollary.

Corollary 1.1. *Let α be a quadratic algebraic number. Then for every natural number n, we have either $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n) \neq 0$ or $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^{n+1}) \neq 0$.*

Hence, in view of Proposition 1.1, it is natural to study the growth of lim sup or lim inf of $|\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)|$.

First, we look at the case when α is a root of unity. The root of unity plays a central role in the theory of algebraic numbers. It can be easily seen that if α is a root of unity then the sequence $\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)\right)_n$ is bounded. In this article, first, we prove that this condition is necessary and sufficient. More precisely, we prove the following.

Theorem 1.1. *Let α be a non-zero algebraic integer. Then α is a root of unity if and only if the sequence $\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)\right)_{n\geq 1}$ is bounded. Moreover, if the sequence $\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)\right)_{n\geq 1}$ is bounded, then it is periodic.*

Theorem 1.1 can be viewed as a generalization of Kronecker's theorem which states that *a non-zero algebraic integer α such that $|\alpha| \leq 1$ and $|\beta| \leq 1$ for all conjugates of α if and only if α is a root of unity.* This because, if $|\beta| \leq 1$ for all conjugates of α, then $|\mathrm{Tr}(\alpha^n)|$ is bounded for all natural numbers n.

The moreover part of Theorem 1.1 is an interesting fact in the following sense. There is a folklore conjecture which says that *the only badly approximable irrational numbers are quadratic irrationals.* It is a well-known fact in continued fraction that *if $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$ has an infinite continued fraction expansion with $a_m \leq M$ for all $m \geq 0$ and for some $M > 0$, then α is a badly approximable number.* In order to prove the folklore conjecture, it is enough to prove that *if $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$ is an infinite continued fraction such that $a_m \leq M$ for all $m \geq 0$, then the sequence $(a_n)_n$ is periodic.*

We can re-interpret the moreover part of Theorem 1.1 as follows. Suppose for some algebraic integer α if $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n) := a_n$, and if $a_n \geq 0$ for all n or $a_n \leq 0$ for all n and bounded, then since $a_n \in \mathbb{Z}$, the infinite continued fraction $\beta = [a_1, \ldots, a_n, \ldots]$ is a quadratic irrational. This in particular says that the bounded sequence (a_n) is periodic. It is a general problem that *if a power series $f(x) := \sum_{n=1}^{\infty} a_n x^n \in \mathbb{Z}[[x]]$ with bounded sequence (a_n), then whether $f(x)$ is a rational function or not?* We refer to a recent result of Bell, Nguyen and Zannier [1] for more information.

By Theorem 1.1, if α is a non-zero algebraic integer which is not a root of unity, then the sequence $\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)\right)_{n\geq 1}$ is unbounded. If $\alpha > 1$ is a Pisot-Vijayaraghavan number (means, it is an algebraic integer and if β is any other conjugate of α, then $|\beta| < 1$), then for any $\epsilon > 0$, there exists N such that for all $n \geq N$, we have

$$\alpha^n - \epsilon \leq \mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n) \leq \alpha^n + \epsilon.$$

If α is any non-zero algebraic integer, then for all $n \geq 1$, we have

$$\frac{|\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)|}{\mathrm{Hou}(\alpha)^n} < \infty,$$

where $\mathrm{Hou}(\alpha) = \max\{|\beta| \mid \beta \text{ is a conjugate of } \alpha\}$. Since $\mathrm{Hou}(\alpha)$ is a constant, in the following theorem, we study the growth of $\mathrm{Tr}(\alpha^n)$ by replacing $\mathrm{Hou}(\alpha)^n$ by $C^{f(n)}$ for some positive constant C, where $f(n)$ is a sub-linear function of n. First, we define a sub-linear function as follows.

Definition 1.1. *A function $f : \mathbb{N} \to (0, \infty)$ is said to be a sub-linear function, if $\lim_{n\to\infty} \frac{f(n)}{n} = 0$.*

Now, we state our result as follows.

Theorem 1.2. *Let α be a non-zero algebraic integer that is not a root of unity and f is a given sub-linear function on $\mathbb{N}$. Then, the following statements are true;*

(1) *We have*

$$\limsup_{n\to\infty} \frac{|\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)|}{e^{f(n)}} = +\infty. \tag{1.1}$$

(2) *Either*

$$\liminf_{n\to\infty} \mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n) = 0$$

or

$$\liminf_{n\to\infty} \frac{|\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)|}{e^{f(n)}} = +\infty. \tag{1.2}$$

We would end the introduction by pointing out another connection of the main Theorem 1.2 in a geometric setting, which would also give a sequence of examples where Theorem 1.2 is true. For a fixed integer $Q > 1$, an algebraic number $\alpha \in \overline{\mathbb{Q}}$ is called a *Q-Weil number*, if for every embedding $\imath : \mathbb{Q} \to \mathbb{C}$, one has $|\imath(\alpha)|_{\mathbb{C}} = Q^{\frac{1}{2}}$ where $|.|_{\mathbb{C}}$ is the usual complex absolute value. A Q-Weil number is called *integral* if it is also an algebraic integer. One then defines $A(k) := \sum_{i=1}^{n} \alpha_i^k$ for given $\alpha_1, \alpha_2, \ldots, \alpha_n$ integral Q-Weil numbers. Now from a corollary of Evertse (Corollary 2, [7]), one gets that there exists a real constant $C_1 > 0$ for a given real number $\epsilon > 0$, such that for any integer $k \geq 1$, either $A(k) = 0$ or for any archimedean absolute value on $\overline{\mathbb{Q}}$, one has $|A(k)| \geq C_1 Q^{k(1-\epsilon)}$. As Q-Weil numbers are not roots of unity, this result is in the setup of Theorem 1.2.

More specifically, one can apply this for $X/\mathbb{F}_q$, a proper smooth variety over a finite field $\mathbb{F}_q$ with $q = p^k$ for some prime number p and some natural number k. Fix an integer $i \geq 1$, and a prime $\ell \neq p$. Consider $A_i(n) := \mathrm{Trace}(Frob_q^n | H^i_{\text{ét}}(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell))$, where $n \geq 1$ is an integer, $Frob_q$ is the Frobenius automorphism acting on the étale cohomology groups. By Deligne's proof of Weil conjectures, the eigenvalues of $Frob_q$ on H^i are integral q^i-Weil numbers. Then applying the previous corollary from Evertse, one gets for all sufficiently large n, either $A_i(n) = 0$ or $|A_i(n)| \geq q^{\frac{in(1-\epsilon)}{2}}$, for a real number $\epsilon > 0$. Indeed, if one takes an ordinary elliptic curve over $\mathbb{F}_q$, then there are only two Frobenius eigenvalues (say, α and $\overline{\alpha}$) which are conjugates of each other and integral q-Weil numbers. Then for $A(n) := \alpha^n + \overline{\alpha}^n$, using Baker-Wüstholz theorem [5], (Corollary 4.3, [4]), one gets that for all $n \geq 1$, the explicit archimedean lower bound $|A(n)| \geq \frac{1}{\pi} \cdot q^{\frac{n}{2} - 2^{39}\log(2n)}$ holds. Essentially, this provides a supply of examples where Theorem 1.2 is true and it shows that Theorem 1.2 generalizes these previous results for arbitrary algebraic integers which are not roots of unity, not only for Q-Weil numbers, but the bound is less explicit.

2. Preliminaries

Let $K \subset \mathbb{C}$ be a number field which is Galois over $\mathbb{Q}$ with Galois group $\mathrm{Gal}(K/\mathbb{Q})$. Let M_K be the set of all inequivalent places of K and M_∞ be the set of all archimedean places of K. For each place $v \in M_K$, we denote $|\cdot|_v$ the absolute value corresponding to v, normalized with respect to K. Indeed if $v \in M_\infty$, then there exists an automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ of K such that for all $x \in K$,

$$|x|_v = |\sigma(x)|^{d(\sigma)/[K:\mathbb{Q}]},$$

where $d(\sigma) = 1$, if $\sigma(K) = K \subset \mathbb{R}$; and $d(\sigma) = 2$ otherwise. Note that since K is Galois over $\mathbb{Q}$, the function $d(\sigma)$ is constant. Non-archimedean absolute values are normalized accordingly so that the product formula $\prod_{\omega \in M_K} |x|_\omega = 1$ holds true for any $x \in K^\times$.

For all $x \in K$, the *absolute Weil height*, $H(x)$, is defined as

$$H(x) := \prod_{\omega \in M_K} \max\{1, |x|_\omega\}.$$

For a vector $\mathbf{x} = (x_1, \ldots, x_n) \in K^n$ and for a place $\omega \in M_K$, the ω-norm for $\mathbf{x}$ denote by $||\mathbf{x}||_\omega$ and defined by

$$||\mathbf{x}||_\omega := \max\{|x_1|_\omega, \ldots, |x_n|_\omega\}$$

and the *projective height*, $H(\mathbf{x})$, is defined by

$$H(\mathbf{x}) = \prod_{\omega \in M_K} ||\mathbf{x}||_\omega \text{ for all } \mathbf{x} \in K^n.$$

In order to prove Theorems 1.1 and 1.2, we also need the following theorem proved in [10, Proposition 2.3]. This result is an application of the Schmidt-Subspace Theorem.

Theorem 2.1. *(A. Kulkarni et al. [10]) Let K be a number field which is Galois over $\mathbb{Q}$ and S be a finite set of places, containing all the archimedean places. Let $\lambda_1, \ldots, \lambda_k$ be non-zero elements of K. Let $\varepsilon > 0$ be a positive real number and $\omega \in S$ be a distinguished place. Let $\mathfrak{E}$ be the set of all $(u_1, \ldots, u_k, b_1, \ldots, b_k) \in (\mathcal{O}_S^\times)^k \times K^*$ which satisfies the inequality*

$$0 < \left| \sum_{j=1}^{k} \lambda_j b_j u_j \right|_\omega \leq \frac{\max\{|b_1 u_1|_\omega, \ldots, |b_k u_k|_\omega\}}{\left(\prod_{i=1}^{k} H(b_j)\right)^{k+1+\varepsilon} H(u_1, \ldots, u_k, 1)^\varepsilon}, \tag{2.1}$$

where $\mathcal{O}_S^\times$ is the ring S-units in K. If $\mathfrak{E}$ is an infinite subset, then there exists $c_1, \ldots, c_k \in K$, not all zero, such that

$$c_1 u_1 + \cdots + c_k u_k = 0$$

holds true for infinitely many elements $(u_1, \ldots, u_k)$ of $\mathfrak{E}$.

3. Proof of Propositions 1.1 and 1.2

Proof of Proposition 1.1. Let $\alpha \neq 0$ be a given algebraic integer. Suppose $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^m) = 0$ for all $m \geq N$. Let $f(x) \in \mathbb{Z}[x]$ be the minimal polynomial of α of degree d. Consider the polynomial $g(x) = \prod_{i=1}^{d}(1 - \alpha_i x) \in \mathbb{Q}[x]$ where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ are the Galois conjugates of α. Note that $\alpha_1^{-1}, \ldots, \alpha_d^{-1}$ are all the roots of $g(x)$. Consider the logarithmic derivative of $g(x)$ to get

$$\frac{g'(x)}{g(x)} = \sum_{i=1}^{d} \frac{-\alpha_i}{1 - \alpha_i x} = -\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha) - \mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^2)x - \cdots - \mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)x^n - \cdots. \tag{3.1}$$

By the assumption $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^m) = 0$ for all $m \geq N$. Hence, RHS of (3.1) is a polynomial $h(x)$ of degree $\leq N - 1$ and hence we get $g'(x) = g(x)h(x)$, a contradiction to degree of g' is $\leq d - 1$. Therefore, there exist infinitely many n such that $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n) \neq 0$. □

Proof of Proposition 1.2 Let a be a given natural number. If possible, we assume that $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^m) = 0$ for all $m = a, a+1, \ldots, a+d-1$. Let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ be the Galois conjugates of α. Then we get

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_d \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{d-1} & \alpha_2^{d-1} & \cdots & \alpha_d^{d-1} \end{pmatrix} \begin{pmatrix} \alpha_1^a \\ \alpha_2^a \\ \vdots \\ \alpha_d^a \end{pmatrix} = \begin{pmatrix} \mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^a) \\ \mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^{a+1}) \\ \vdots \\ \mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^{a+d-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the $d \times d$ matrix $A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_d \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{d-1} & \alpha_2^{d-1} & \cdots & \alpha_d^{d-1} \end{pmatrix}$ is Vandermonde, it is invertible and we get $\alpha_i^a = 0$ for all $i = 1, 2, \ldots, d$ which is a contradiction. Hence the proposition. $\square$

4. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Since α is a root of unity, there exists a positive integer h such that $\alpha^h = 1$. Since every positive integer n can be expressed as $n = a + hm$ for some $a \in \{0, 1, \ldots, h-1\}$ and some positive integer m, we conclude that the sequence $\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)\right)_n$ takes values inside the finite set $\{\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^a) : 0 \le a \le h-1\}$. Thus the sequence $\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)\right)_n$ is bounded.

For the converse part, we proceed similar to Lemma 4 in [6] (see also [12, Theorem 2]). Let K be the Galois closure of the number field $\mathbb{Q}(\alpha)$. Let h be the order of the torsion subgroup of $K^\times$ and let $G = \mathrm{Gal}(K/\mathbb{Q})$ be the Galois group of K over $\mathbb{Q}$. Let S be a suitable finite subset of M_K containing all the archimedean places such that α is an S-unit and stable under Galois conjugation.

First we observe that if α is a non-zero algebraic integer such that $\alpha^h \in \mathbb{Z}$, then by the assumption that the sequence $\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)\right)_n$ is bounded, one can easily deduce that $\alpha = \pm 1$. Thus we can assume that $[\mathbb{Q}(\alpha^h) : \mathbb{Q}] = d \ge 2$. By Proposition 1.1, there exists infinitely many natural numbers n such that $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n) \ne 0$. Let $\mathcal{A}$ be such a collection of n's. Since $\mathcal{A}$ is infinite, there an integer $r \in \{0, 1, \ldots, h-1\}$ and an infinite set $\mathcal{A}'$ such that $n = r + hm \in \mathcal{A}$ for all $m \in \mathcal{A}'$. Since the sequence $\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)\right)_n$ is bounded, there exists a nonzero integer B satisfying

$$\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^{r+hm}) = B \tag{4.1}$$

holds for all $m \in \mathcal{A}'$, without loss of generality.

Put $F = \mathbb{Q}(\alpha^h)$ and see that $[F : \mathbb{Q}] = d > 1$. Let H be the subgroup of G fixing F and let $G/H = \{\sigma_1 H, \ldots, \sigma_d H\}$, where $\sigma_1, \ldots, \sigma_d$ be the element of G which are a complete set of representatives. Then (4.1) can be written as

$$\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^{r+hm}) = A_1\sigma_1(\alpha^{hm}) + \cdots + A_d\sigma_d(\alpha^{hm}) = B \tag{4.2}$$

holds for all $m \in \mathcal{A}'$, where $A_j = \sigma_j\left(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^r)\right)$ for all $j = 1, 2, \ldots, d$.

Indeed, we prove that if (4.2) holds for infinitely many m's, then α must be a root of unity. Suppose α is not a root of unity. Since B is a nonzero integer, there exists an archimedean place w such that

$$M = \max\{|\sigma_1(\alpha)|_w, \ldots, |\sigma_d(\alpha)|_w\} > 1. \tag{4.3}$$

Choose $\varepsilon < \dfrac{\log M}{\log H(\alpha)}$ and from (4.1) and (4.3), we deduce

$$|A_1\sigma_1(\alpha^{hm}) + \cdots + A_d\sigma_d(\alpha^{hm})|_w = |B|_w \cdot 1 < \frac{\max\{|\sigma_1(\alpha^{hm})|_w, \ldots, |\sigma_d(\alpha^{hm})|_w\}}{H^\epsilon(\alpha^{hm})}$$

for infinitely many positive integers $m \in \mathcal{A}'$. We then apply Theorem 2.1 to get a non-trivial relation

$$c_1\sigma_1(\alpha^{hm}) + \cdots + c_d\sigma_d(\alpha^{hm}) = 0 \tag{4.4}$$

holds for infinitely many $m \in \mathcal{A}'$, where $c_i \in K$ and not all are zero. Now we proceed exactly as in [12, Theorem 2, case $d > 1$] to get a contradiction, which completes the proof of the first part of the theorem.

For the moreover part, assume that the sequence $(\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^m))_m$ is bounded. Then by the first part, we conclude that α is a root of unity and hence the trace of α^m are periodic. □

Proof of Theorem 1.2. Let $L = \mathbb{Q}(\alpha)$ be the number field of degree d and K be its Galois closure. Let h be the order of the torsion subgroup of $K^\times$ and let $G = \mathrm{Gal}(K/\mathbb{Q})$ be the Galois group of K over $\mathbb{Q}$. Let S be a suitable finite subset of M_K containing all the archimedean places such that α is an S-unit and stable under Galois conjugation.

First, we note that if for some power of α is an integer and α is not a root of unity, then (1.1) holds true easily. Therefore we can assume that no integral power of α is an integer. By Proposition 1.1, we know that $\mathrm{Tr}_{L/\mathbb{Q}}(\alpha^n) \neq 0$ for infinitely many positive integers n. Let $\mathcal{B}$ be collection of such n.

Suppose the conclusion (1.1) does not hold. Then there exists an integer $C > 1$ such that

$$0 < |\mathrm{Tr}_{L/\mathbb{Q}}(\alpha^n)| \le Ce^{f(n)} \tag{4.5}$$

holds for all large integers $n \in \mathcal{B}$. Since $\mathcal{B}$ is infinite, there exists $a \in \{0, 1, \ldots, h-1\}$ such that $n = a + hm \in \mathcal{B}$ for all positive integers m. Then the inequality in (4.5) can be read as

$$0 < |\mathrm{Tr}_{L/\mathbb{Q}}(\alpha^{a+hm})| \le Ce^{f(a+hm)} \tag{4.6}$$

for all positive integers m, without loss of generality.

Since α is not a root of unity, there is an archimedean place w such that

$$M = \max\{|\sigma_1(\alpha)|_w, \ldots, |\sigma_d(\alpha)|_w\} > 1. \tag{4.7}$$

Choose $\varepsilon < \dfrac{\log M}{\log H(\alpha)}$. Let $F = \mathbb{Q}(\alpha^h)$ be the number field whose degree $[F : \mathbb{Q}] = d > 1$, by the assumption. Let H be the subgroup of G fixing F and let $G/H = \{\sigma_1 H, \ldots, \sigma_d H\}$, where $\sigma_1, \ldots, \sigma_d$ be the element of G which are a complete set of representatives. From (4.6) and (4.7), we get

$$|\mathrm{Tr}_{L/\mathbb{Q}}(\alpha^n)|_w \le Ce^{f(n)} \cdot 1 < \frac{\max\{|\sigma_1(\alpha^n)|_w, \ldots, |\sigma_d(\alpha^n)|_w\}}{H^\varepsilon(\alpha^n)}$$

for all sufficiently large values of $n \in \mathcal{B}$ of the form $n = a + hm$, as $f(n)/n \to 0$ as $n \to \infty$. By Theorem 2.1, we arrive at a non-trivial relation of the form

$$c_1\sigma_1(\alpha^{hm}) + \cdots + c_d\sigma_d(\alpha^{hm}) = 0, \quad c_i \in K$$

holds for all large values of m, where $n = a + hm$. We then conclude the proof of the theorem exactly as in Theorem 1.1. This proves both parts of the theorem. □

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