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ON A THEOREM OF MAHLER

N. K. MEHER, K. SENTHIL KUMAR, AND R. THANGADURAI

(Communicated by Matthew A. Papanikolas)

Dedicated to Michel Waldschmidt on his 70th birthday

ABSTRACT. Let $b \geq 2$ be an integer and α a non-zero real number written in base b. In 1973, Mahler proved the following result: *Let α be an irrational number written in base b and let $n \geq 1$ be a given integer. Let $B = b_0 b_1 \dots b_{n-1}$ be a given block of digits in base b of length n. Then, there exists an integer X with $1 \leq X < b^{2n+1}$ such that B occurs infinitely often in the base b representation of the fractional part of $X\alpha$.* In this short note, we deal with some conditional quantitative version of this result.

1. Introduction

Let $b \geq 2$ be an integer. We say a non-zero real number α is *written in base b* if there exist $a_0 \in \mathbb{Z}$ and non-negative integers $a_1, a_2, \dots, a_k$, with $0 \leq a_k \leq b-1$ such that

$$\alpha = a_0 + \frac{a_1}{b} + \frac{a_2}{b^2} + \cdots + \frac{a_k}{b^k} + \cdots . \tag{1.1}$$

A basic result says that every real number has a base b representation and this way of representing real numbers provides a necessary and sufficient condition for it to be a rational number. More precisely, we know that a real number α has an eventually periodic base b representation (this means that $a_{n+k} = a_n$ for all $n \geq N_0$ and for some integers $N_0 \geq 1$ and $k \geq 1$) if and only if it is a rational number. In this article, we shall deal with irrational numbers.

Since we shall study the digits which appear after the decimal place, we need to deal with the fractional part of α (denoted by $\{\alpha\}$) of real numbers. Hence, we shall assume that $\alpha \in [0,1)$ is an irrational number. Let $B = b_0 b_1 \dots b_{k-1}$ be a given block of digits of length k in base b. For any real number $\alpha \in [0,1)$ satisfying (1.1) and for any integer $m \geq 1$, we let

$$N_B(m, \alpha) := |\{i \ : \ a_i a_{i+1} \dots a_{i+k-1} = B, \text{ for any } i \leq m-k+1\}| \tag{1.2}$$

denote the number of times the given block B occurs in the first m digits of α in base b.

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The given block B of length k in base b is said to occur in the base b expansion of a real number α satisfying (1.1) with the *frequency* ν if

$$\lim_{m\to\infty} \frac{N_B(m,\alpha)}{m} = \nu. \tag{1.3}$$

A real number α is said to be *normal to base* b if for every integer $k \geq 1$, every block B of length k occurs in α with the frequency $1/b^k$.

In 1909, E. Borel [3] proved that with respect to Lebesgue measure almost all real numbers are normal in all the bases $b \geq 2$ and conjectured the following.

Conjecture 1 (E. Borel, [3]). *Every algebraic irrational must be normal to base b for every integer $b \geq 2$.*

No single example of an algebraic irrational is known to satisfy Borel's conjecture yet. Even for the simplest algebraic irrational number $\sqrt{2}$, we do not know whether it is normal to the base 2. In order to prove Borel's conjecture, we need to prove that if α is an algebraic irrational satisfying (1.1) and for a given block $B = b_0 b_1 \dots b_{k-1}$ in base b, then we have $N_B(m,\alpha) \sim m/b^k$ for all large enough integers m.

In 1973, Mahler [4] proved that *if α is an irrational number satisfying* (1.1) *and B is a given block in base b of length k (for a given integer $k \geq 1$), then there exists an integer X with $1 \leq X < b^{2k+1}$ such that the block B occurs in the base b representation of the fractional part of $X\alpha$ infinitely often.* In particular, if we take α to be algebraic irrational, then there exists an integer X with $1 \leq X < b^{2k+1}$ such that the base b representation of $\{X\alpha\}$ captures the given pattern B infinitely often. It may happen that for a different B, we might get different X. Hence, finding one example of an algebraic irrational α which captures all patterns of length k for all integers $k \geq 1$ infinitely often is not solved yet. In 1994, Berend and Boshernitzan [2] improved the bound of the integer X in the theorem of Mahler. Indeed, they proved the same result with $1 \leq X < 2b^{k+1}$. In 2008, Adamczewski and Rampersad [1] proved that every algebraic number contains infinitely many occurrences of 7/3-powers in its binary expansion. In the same paper, they proved that every algebraic number contains either infinitely many occurrences of squares or infinitely many occurrences of one of the blocks 010 or 02120 in its ternary expansion.

If $B = b_0 \dots b_{k-1}$ is a given block in base b, then we write the integer $b_0 b^{k-1} + b_1 b^{k-2} + \cdots + b_{k-1}$ associated to B by $L(B)$. Note that if the block $B = \underbrace{0\dots0}_{\ell\ times} b_\ell \dots b_{k-1}$ for some integer $\ell \leq k-1$ and $b_\ell \neq 0$, then the integer $L(B) \leq b^{k-1-\ell} - 1$. However, we treat this integer $L(B) \leq b^{k-1} - 1$ without getting into much finer analysis.

In this short note, we are dealing with a conditional quantitative version of Mahler's Theorem. Indeed, we prove the following theorems.

Theorem 1. *Let $\alpha \in [0,1)$ be an irrational number satisfying* (1.1). *Let m and n be positive integers satisfying $1 \leq m \leq n$ and let $B = b_0 b_1 \dots b_{m-1}$ be a given block of digits in base b. If the block $B_0 = \underbrace{00\dots0}_{n\ times} a$ for some integer $a \in \{1, 2, \dots, b-1\}$ occurs in the base b representation of α with the frequency ν for some real number ν satisfying $0 < \nu \leq 1$, then there exists an integer X with $L(B) < X < b(L(B)+1)$ such that B occurs in the base b representation of the fractional part of $X\alpha$ with the frequency at least $\frac{\nu}{b^{m+1}}$, where $L(B)$ is the non-negative integer associated to B.*

When B_0 occurs in α with the frequency b^{-n-1}, then B occurs in $X\alpha$ with the frequency at least b^{-m-n-2} for some positive integer X. We shall improve Theorem 1 under additional assumptions as follows.

Theorem 2. *Let $\alpha \in [0,1)$ be an irrational number satisfying* (1.1). *Let $B = b_0 b_1 \dots b_{n-1}$ be a given block in base b for some integer $n \geq 1$. Suppose the block $B_0 = \underbrace{00\dots0}_{n \text{ times}} c_1 \dots c_{n+2}$ with $c_j \in \{0,1,\dots,b-1\}$ for all $j = 1,\dots,n+2$ and $c_1 \neq 0$ occurs in α with the frequency ν for some real number ν satisfying $0 < \nu \leq 1$. Then there exists an integer X with $L(B) < X < b(L(B)+1)$ such that the given block B occurs in the fractional part of $X\alpha$ with frequency at least $\dfrac{\nu}{b+1}$.*

Corollary 1. *Let $\alpha \in [0,1)$ be an irrational number satisfying* (1.1). *Let $B = b_0 b_1 \dots b_{n-1}$ be a given block in base b for some integer $n \geq 1$. Suppose the block $B_0 = \underbrace{00\dots0}_{n \text{ times}} 1 \underbrace{0\dots0}_{n+1 \text{ times}}$ occurs in α with the frequency ν for some real number ν satisfying $0 < \nu \leq 1$. Then there exists an integer X for some $X \in \{k, k+1, \dots, k+b\}$ with $k = \left[\dfrac{b^{n+2}}{b^{n+1}+1} L(B)\right] + 1$ such that the given block B occurs in the fractional part of $X\alpha$ with the frequency at least $\dfrac{\nu}{b+1}$.*

For example, let us take the Champernowne number $\alpha = 0.1234567891011\dots$ written in base 10. Then it is well known that α is irrational and normal to the base 10. For any integer $n \geq 1$, take the block $B = 1\underbrace{00\dots0}_{n \text{ times}}1$ in base 10. Since the block $B_0 = \underbrace{00\dots0}_{n+2 \text{ times}} 1 \underbrace{00\dots0}_{n+3 \text{ times}}$ in base 10 occurs in the base 10 representation of α with the frequency $\dfrac{1}{10^{2n+6}}$, by Corollary 1, there exists an integer X satisfying $X \in \{10^{n+2}+1, \dots, 10^{n+2}+11\}$ such that the block B occurs in the base 10 representation of $\{X\alpha\}$ with frequency at least $\dfrac{1}{11 \times 10^{2n+6}}$, as $L(B) = 10^{n+1}+1$.

2. Preliminaries

We shall start with the following main lemma. The proof of this lemma is essentially due to Mahler [4], and we are reformulating in the following way.

Lemma 2.1. *Let $\alpha \in [0,1)$ be an irrational number satisfying* (1.1). *For any integer $n \geq 1$ and m with $1 \leq m \leq n$, let $B = b_0 b_1 \dots b_{m-1}$ be a given block in base b. Suppose the block $B_0 = \underbrace{00\dots0}_{n \text{ times}} a$ for some integer $a \in \{1,\dots,b-1\}$ occurs in the base b representation of α, say, $a_{h_0} = a_{h_0+1} = \cdots = a_{h_0+n-1} = 0$ for some integer $h_0 \geq 1$. Then there exists a positive integer X with $L(B) < X < b(L(B)+1)$ such that the given block B occurs in the base b representation of $\{X\alpha\}$ with*

$$c_{h_0+n-m} = b_0, c_{h_0+n-m+1} = b_1, \dots, c_{h_0+n-1} = b_{m-1},$$

where $\{X\alpha\} = \displaystyle\sum_{h=1}^{\infty} \frac{c_h}{b^h}$.

Proof. Suppose that the block $B_0 := \underbrace{00\ldots0}_{n\ times}a$ with $a \in \{1, \ldots, b-1\}$ occurs in the base b representation of α such that

$$a_{h_0} = a_{h_0+1} = \cdots = a_{h_0+n-1} = 0, \text{ and } a_{h_0+n} = a$$

for some integer $h_0 \geq 1$. Putting $h_1 = h_0 + n$, we get $a_{h_1} = a$. Now, we define

$$s = \sum_{h=1}^{h_0-1} \frac{a_h}{b^h} \text{ and } t = \sum_{h=h_1}^{\infty} \frac{a_h}{b^{h-h_1+1}} \tag{2.1}$$

and hence we get

$$\alpha = \{\alpha\} = s + \frac{t}{b^{h_1-1}}. \tag{2.2}$$

Also, note that $s \in \mathbb{Q}$ and t is an irrational number. Indeed,

$$t = \sum_{h=h_1}^{\infty} \frac{a_h}{b^{h-h_1+1}} > \frac{a_{h_1}}{b^{h_1-h_1+1}} \geq \frac{1}{b},$$

as $a_{h_1} \geq 1$, and hence we get

$$\frac{1}{b} < t < 1. \tag{2.3}$$

We write the integer associated to the given block $B = b_0 b_1 \ldots b_{m-1}$ in base b by $L(B) = b_0 b^{m-1} + b_1 b^{m-2} + \cdots + b_{m-1} \neq 0$. (If $L(B) = 0$, then the lemma follows, by taking $X = 1$.) By (2.3), we see that there exists an integer X such that $Xt \in (L(B), L(B)+1)$. Then, we get:

(1) $[Xt] = L(B)$, the integral part of Xt.

(2) Since $Xt \in (L(B), L(B)+1)$, we get

$$\frac{L(B)}{t} < X < \frac{L(B)+1}{t}, \tag{2.4}$$

and hence, by (2.3), we get $L(B) < X < b(L(B)+1)$.

Now, note that

$$\frac{Xt}{b^{h_1-1}} = \frac{b_0}{b^{h_1-m}} + \frac{b_1}{b^{h_1-m+1}} + \cdots + \frac{b_{m-1}}{b^{h_1-1}} + \frac{\{Xt\}}{b^{h_1-1}}. \tag{2.5}$$

Therefore, by (2.2) and (2.5), we get

$$X\{\alpha\} = Xs + \frac{Xt}{b^{h_1-1}} = Xs + \frac{b_0}{b^{h_1-m}} + \frac{b_1}{b^{h_1-m+1}} + \cdots + \frac{b_{m-1}}{b^{h_1-1}} + \frac{\{Xt\}}{b^{h_1-1}}. \tag{2.6}$$

We have

$$\{X\{\alpha\}\} = \{X\alpha\}, \tag{2.7}$$

because

$$X\{\alpha\} - \{X\alpha\} = X(\alpha - [\alpha]) - (X\alpha - [X\alpha]) = [X\alpha] - X[\alpha] \in \mathbb{Z}.$$

Also, since X is an integer, by (2.1), we conclude that the denominator of Xs divides b^{h_0-1}. Since $h_0 - 1 < h_1 - m$ and by (2.7), we get

$$\{X\alpha\} = \frac{d_1}{b} + \cdots + \frac{d_{h_0-1}}{b^{h_0-1}} + \frac{b_0}{b^{h_1-m}} + \frac{b_1}{b^{h_1-m+1}} + \cdots + \frac{b_{m-1}}{b^{h_1-1}} + \frac{\{Xt\}}{b^{h_1-1}},$$

for some $d_i \in \{0, 1, \ldots, b-1\}$. Note that Xt is an irrational number and hence the base b representation of $\frac{\{Xt\}}{b^{h_1-1}} = \sum_{h=h_1}^{\infty} \frac{c_h}{b^h}$. Therefore, as $h_1 = h_0 + n$, we see that

$$h_1 - m = h_0 + n - m, h_1 - m + 1 = h_0 + n - m + 1, \ldots, h_1 - 1 = h_0 + n - 1$$

and hence we have proved the lemma. □

Lemma 2.2. *Let $L > 1$ be an integer written in base b as $b_0 b_1 \ldots b_{m-1}$ with $b_0 \neq 0$ for some positive integer m. Let S be the set of all positive integers k for which there exists an irrational number t satisfying $1/b < t < 1$ and $kt \in (L, L+1)$. Then, we have $|S| < b^{m+1}$.*

Proof. Let $k \in S$ be any positive integer. Then there exists an irrational number t with $1/b < t < 1$ such that $kt \in (L, L+1)$. Therefore, we get $L < kt < L+1$, which implies that

$$L < \frac{L}{t} < k < \frac{L+1}{t} < b(L+1).$$

Since $b(L+1) - L \leq bb^m - 1 < b^{m+1}$, we see that $|S| < b^{m+1}$, as desired. □

Corollary 2.2.1. *Let $\alpha \in [0,1)$ be an irrational number satisfying* (1.1). *For any integers $n \geq 1$ and m with $1 \leq m \leq n$, let $B = b_0 b_1 \ldots b_{m-1}$ be a given block in base b. If the block $B_0 = \underbrace{0 \ldots 0}_{n\ times} a$ for some $a \in \{1, 2, \ldots, b-1\}$ occurs infinitely often in the base b representation of α, then there exists an integer X with $L(B) < X < b(L(B)+1)$ such that the given block B occurs in the fractional part of $X\alpha$ infinitely often.*

Proof. Suppose that the block $B_0 = \underbrace{0 \ldots 0}_{n\ times} a$ for some $a \in \{1, 2, \ldots, b-1\}$ occurs infinitely often in the base b representation of $\alpha = 0.a_1 a_2 \ldots a_h \ldots$. Let the block $B_0 = \underbrace{00 \ldots 0}_{n\ times} a$ occur in the base b representation of α as follows:

$$\begin{aligned} \alpha \quad = \quad & 0.a_1 a_2 \ldots a_h \ldots a_{h_0^{(1)}-1} \underbrace{00 \ldots 0}_{n\ times} a a_{h_0^{(1)}+n+1} \ldots a_{h_0^{(2)}-1} \underbrace{00 \ldots 0}_{n\ times} a a_{h_0^{(2)}+n+1} \\ & \ldots a_{h_0^{(r)}-1} \underbrace{00 \ldots 0}_{n\ times} a a_{h_0^{(r)}+n+1} \ldots . \end{aligned}$$

For any integer $r = 1, 2, \ldots$, we let

$$s_r = 0.a_1 \ldots a_{h_0^{(r)}-1} \text{ and } t_r = 0.a a_{h_0^{(r)}+n+1} \ldots .$$

Clearly, s_r is a rational number and t_r is an irrational number satisfying

$$\alpha = s_r + \frac{t_r}{b^{h_0^{(r)}+n-1}}.$$

For each pair (s_r, t_r), we apply Lemma 2.1 to get an integer X_r with $L(B) < X_r < b(L(B)+1)$ such that the given block B occurs in the fractional part of $X_r \alpha$. By Lemma 2.2, there can be at most b^{m+1} distinct integers X_r available in the interval $(L(B), b(L(B)+1))$. The assumption that B_0 occurs infinitely often together with the Dirichlet box principle implies that there is an integer $X \in (L(B), b(L(B)+1))$ for which the assertion is true. □

We improve the estimate for $|S|$ under some extra assumptions as follows.

Lemma 2.3. *Let $L > 1$ be an integer written in base b as $b_0 b_1 \dots b_{m-1}$ for some integer $m \geq 1$. Let $c_1, c_2, \dots, c_{m+2}$ be given integers such that $c_1 \neq 0$ and $c_i \in \{0, 1, \dots, b-1\}$ for all $i = 1, 2, \dots, m+2$. Let S be the set of all positive integers k for which there exists an irrational number t written in base b as $0.c_1 c_2 \dots c_{m+2} d_1 d_2 \dots$ with $d_i \in \{0, 1, \dots, b-1\}$ and $kt \in (L, L+1)$. Then*

$$S \subset \{X, X+1, \dots, X+b\}$$

for some positive integer X. In particular, $|S| \leq b+1$.

Proof. Clearly, $|S| \geq 1$, and as any irrational number t of the form $t = 0.c_1 \dots c_{m+2} d_1 \dots$ with $c_1 \neq 0$ satisfies $1/b < t < 1$, we see that there exists an integer $k \geq 1$ such that $kt \in (L, L+1)$. We now prove the upper bound.

In order to prove $|S| \leq b+1$, we take $m_1 \neq m_2$ in S and we prove that $|m_1 - m_2| < b+1$.

Take $t_1 = 0.c_1 c_2 \dots c_{m+2} d_1 d_2 \dots$ and $t_2 = 0.c_1 c_2 \dots c_{m+2} e_1 e_2 \dots$ are two irrational numbers written in base b. Then, we have

$$|t_1 - t_2| \leq \frac{|d_1 - e_1|}{b^{m+3}} + \frac{|d_2 - e_2|}{b^{m+4}} + \cdots \leq \frac{b-1}{b^{m+3}} \frac{b}{b-1} = \frac{1}{b^{m+2}}. \tag{2.8}$$

Let m_1 and m_2 be two distinct positive integers in S. Then, there exist two irrational numbers t_1 and t_2 satisfying (2.8) and $m_i t_i \in (L, L+1)$ for all $i = 1, 2$. Therefore,

$$|m_1 t_1 - m_2 t_2| < 1. \tag{2.9}$$

Since $m_1 \neq m_2$, without loss of generality, we may assume that $m_1 > m_2 > 1$.

If $t_1 = t_2$, then $|m_1 t_1 - m_2 t_2| = t_1 |m_1 - m_2| < 1$ implies that $|m_1 - m_2| < 1/t_1 < b$, as $t_1 > 1/b$. In this case, we are done. Hence we can assume that $t_1 \neq t_2$. If $t_1 > t_2$, then, clearly, as $m_1 > m_2$, we see that

$$0 < (m_1 - m_2) t_2 = m_1 t_2 - m_2 t_2 < m_1 t_1 - m_2 t_2 < 1 \implies m_1 - m_2 < 1/t_2 < b,$$

as desired.

Thus, we assume that $m_1 > m_2$ and $t_1 < t_2$ together with $m_1 t_1, m_2 t_2 \in (L, L+1)$.

We claim that $m_1 t_1 > m_2 t_2$. If possible, we suppose $m_1 t_1 \leq m_2 t_2$. We consider

$$(m_2 - m_2 + m_1) t_1 = m_1 t_1 \leq m_2 t_2 \implies (m_1 - m_2) t_1 \leq m_2 (t_2 - t_1).$$

Since $t_2 - t_1 > 0$, by (2.8), we get

$$(m_1 - m_2) t_1 \leq \frac{m_2}{b^{m+2}} \implies (m_1 - m_2) \leq \frac{m_2}{t_1 b^{m+2}}.$$

Since $1/b < t_i$ for $i = 1, 2$ and by (2.4), $m_2 < \dfrac{L+1}{t_2} < b(L+1)$, we get

$$m_1 - m_2 < \frac{b^2 (L+1)}{b^{m+2}} = \frac{L+1}{b^m}. \tag{2.10}$$

Since $L \leq b^m - 1$, we get $L + 1 \leq b^m$ and hence, by (2.10), we get $m_1 - m_2 < 1$, which is a contradiction to the fact that $m_1 \geq m_2 + 1$. Hence, we always have $m_1 t_1 > m_2 t_2$.

Since $m_1 t_1 > m_2 t_2$, we have $0 < m_1 t_1 - m_2 t_2 < 1$ and hence $-1 < m_2 t_2 - m_1 t_1 < 0$. Thus, we get

$$m_2 t_2 > -1 + m_1 t_1 = -1 + (m_1 - m_2 + m_2) t_1 = -1 + (m_1 - m_2) t_1 + m_2 t_1.$$

Therefore, we have

$$(m_1 - m_2)t_1 < 1 + m_2(t_2 - t_1) \implies m_1 - m_2 < \frac{1}{t_1} + \frac{m_2}{t_1 b^{m+2}},$$

by (2.8). Since $t_1 > 1/b$ and $m_2 b \le b^{m+2}$, we get

$$m_1 - m_2 < b + 1,$$

as desired. This proves the lemma. ☐

Lemma 2.4. *Let $L > 1$ be an integer written in base b as $b_0 b_1 \dots b_{m-1}$ for some integer $m \ge 1$. Let S be the set of all positive integers k for which there exists an irrational number t written in base b as $0.1\underbrace{0\dots0}_{m+1\ times}d_1 d_2 \dots$ with $d_i \in \{0, 1, \dots, b-1\}$ and $kt \in (L, L+1)$. Then*

$$S \subset \{X, X+1, \dots, X+b\}$$

where the integer X is $X = \left[\frac{b^{m+2}}{b^{m+1}+1}L\right] + 1$.

Proof. Put $c_1 = 1$ and $c_2 = \cdots = c_{m+2} = 0$ in Lemma 2.3. Then a positive integer $k \in S$ if and only if there exists an irrational number t written in base b as

$$0.1\underbrace{0\dots0}_{m+1\ times}d_1 d_2 \dots \text{ with } d_j \in \{0, 1, \dots, b-1\}$$

such that $kt \in (L, L+1)$. Note that such an irrational number t satisfies

$$\frac{1}{b} < t < \frac{1}{b} + \frac{b-1}{b^{m+3}} + \frac{b-1}{b^{m+4}} + \cdots = \frac{1}{b} + \frac{1}{b^{m+2}} = \frac{b^{m+1}+1}{b^{m+2}}.$$

Therefore, $k \in S$ if and only if $\frac{L}{t} < k < \frac{L+1}{t}$, and hence we get

$$k \in S \iff \frac{b^{m+2}}{b^{m+1}+1}L < k < b(L+1).$$

Note also that

$$\begin{aligned} b(L+1) - \frac{b^{m+2}}{b^{m+1}+1}L &= \frac{1}{b^{m+1}+1}\left(b^{m+2}L + b^{m+2} + bL + b - b^{m+2}L\right) \\ &= \frac{bL}{b^{m+1}+1} + b < b+1, \end{aligned}$$

as $L \le b^m - 1$. Therefore, by Lemma 2.3, we arrive at the result. ☐

3. Proofs of Theorems 1 and 2 and Corollary 1

Proof of Theorem 1. Suppose that $\alpha \in [0, 1)$ is an irrational number satisfying (1.1) written in base b. By hypothesis, the block $\underbrace{00\dots0}_{n\ times}a$ (for a fixed $a \in \{1, \dots, b-1\}$) occurs in the base b representation of α with the frequency ν for some $0 < \nu \le 1$.

By keeping the same notation as in the proof of Corollary 2.2.1, we see that for each pair (s_r, t_r), by Lemma 2.1, we get an integer X_r satisfying $L(B) < X_r < b(L(B)+1)$ such that the given block B occurs in the base b representation of the fractional part of $X_r \alpha$.

Let S_1 be the subset of the natural numbers which consists of all the natural numbers X_r such that B occurs in the base b representation of the fractional part of $X_r\alpha$ as in Lemma 2.1. By Lemma 2.2, we know that $1 \leq |S_1| < b^{m+1}$.

In order to finish the proof of this theorem, we need to prove that there exists an integer $X \in S_1$ for which the block B occurs in the base b representation of the fractional part of $X\alpha$ with the frequency $\geq \nu/b^{m+1}$.

On the contrary, we can assume that for every integer $X_k \in S_1$, the block B occurs in the base b representation of the fractional part of $X\alpha$ with the frequency, say, $f_k < \frac{\nu}{b^{m+1}}$. To finish the proof, we need to get a contradiction.

For any natural number R, we define the set $S_2(R)$ which consists of distinct blocks, say, $a_{X,i}a_{X,(i+1)}\cdots a_{X,(i+m-1)}$, in the first R digits of the base b representation of the fractional part of $X\alpha$ for some $X \in S_1$ such that $a_{X,i}a_{X,(i+1)}\cdots a_{X,(i+m-1)} = B$. Then, by Lemma 2.1, we get

$$|S_2(R)| \geq N_{B_0}(R, \alpha). \tag{3.1}$$

Since $\nu > 0$, by (3.1), it follows that $f_k > 0$ for some integer $X_k \in S_1$. Now we define

$$\delta = \min\left\{\delta_k : \delta_k = \frac{\nu}{b^{m+1}} - f_k\right\} \text{ for all } X_k \in S_1. \tag{3.2}$$

Since $f_k < \nu/b^{m+1}$ for all $X_k \in S_1$, we see that $\delta > 0$. Choose a real number $\epsilon = \delta/4 > 0$. Since B_0 occurs with the frequency ν in α, by (1.3), there exists a natural number R_0 such that

$$N_{B_0}(R, \alpha) > R(\nu - \epsilon) \text{ for all integers } R \geq R_0. \tag{3.3}$$

Since b and m are fixed positive integers, we let $\epsilon_1 = \frac{\delta}{4b^{m+1}} > 0$. For this ϵ_1, by (1.3), there exists a natural number R_1 such that

$$N_B(R', \{X_k\alpha\}) < R'f_k + R'\epsilon_1 \text{ for all } R' \geq R_1 \text{ and for all } X_k \in S_1. \tag{3.4}$$

Let R be any natural number satisfying $R \geq \max\{R_0, R_1\}$. Also, by the definition of $S_2(R)$, we have

$$|S_2(R)| \leq \sum_{X_k \in S_1} N_B(R, \{X_k\alpha\}). \tag{3.5}$$

Therefore by (3.1), (3.3), (3.4) and (3.5), we get

$$R\nu - R\epsilon < |S_2(R)| \leq \sum_{X_k \in S_1} (f_k R + R\epsilon_1) = \sum_{X_k \in S_1} \left(\frac{\nu}{b^{m+1}} - \delta_k\right) R + R|S_1|\epsilon_1.$$

Hence, by Lemma 2.2, we get

$$R\nu - R\epsilon < R\nu - R\sum_{X_k \in S_1} \delta_k + R|S_1|\epsilon_1.$$

This is equivalent to saying that

$$-\epsilon < -\sum_{X_k \in S_1} \delta_k + |S_1|\epsilon_1 < -\sum_{X_k \in S_1} \delta_k + \frac{\delta}{4} = -\sum_{X_k \in S_1} \delta_k + \epsilon,$$

and hence we get

$$-2\epsilon = -\frac{\delta}{2} < -\sum_{X_k \in S_1} \delta_k,$$

which is a contradiction to the choice of δ as in (3.2). This proves the theorem. □

Proof of Theorem 2. By the assumption together with Lemma 2.3, we get the result along the same lines as the proof of Theorem 1. □

Proof of Corollary 1. By taking $c_1 = 1$ and $c_2 = \cdots = c_{n+2} = 0$ in Theorem 2 and by Lemma 2.4, we get the corollary. □

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Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad, 211019, India

E-mail address: nabinmeher@hri.res.in

Chennai Mathematical Institute, H1, SIPCOT IT Park, Siruseri, Kelambakkam, Chennai, 603103, India

Current address: National Institute of Science Education and Research, HBNI, P.O. Jatni, Khurda 752050, Odisha, India

E-mail address: senthil@niser.ac.in

Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad, 211019, India

E-mail address: thanga@hri.res.in