

SUFFICIENT CONDITIONS FOR A PROBLEM OF POLYA

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ABSTRACT. Let α be a non-zero algebraic number. Let K be the Galois closure of $\mathbb{Q}(\alpha)$ with Galois group G and $\bar{\mathbb{Q}}$ be the algebraic closure of \mathbb{Q} . In this article, among the other results, we prove the following. *If $f \in \bar{\mathbb{Q}}[G]$ is a non-zero element of the group ring $\bar{\mathbb{Q}}[G]$ and α is a given algebraic number such that $f(\alpha^n)$ is a non-zero algebraic integer for infinitely many natural numbers n , then α is an algebraic integer.* This result generalises the result of Polya [Rend. Circ Mat. Palermo, 40 (1915), pp. 1–16], Corvaja and Zannier [Acta Math. 193 (2004), pp. 175–191] and Philippon and Rath [J. Number Theory 219 (2021), pp. 198–211]. We also prove the analogue of this result for rational functions with algebraic coefficients. Inspired by a result of B. de Smit [J. Number Theory 45 (1993), pp. 112–116], we prove a finite version of the Polya type result for a binary recurrence sequences of non-zero algebraic numbers. In order to prove these results, we apply the techniques of Corvaja and Zannier along with the results of Kulkarni *et al.* [Trans. Amer. Math. Soc. 371 (2019), pp. 3787–3804], which are applications of the Schmidt subspace theorem.

1. INTRODUCTION

We deal with the problem of determining whether a given algebraic number α is an algebraic integer under certain conditions. In 1915, Polya had proved the following statement: *If $\text{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n)$ are integers for all natural numbers n , then α is an algebraic integer.* A proof of the above result, by elementary manipulations, does not seem plausible owing to Newton’s identities [8]. In the proof provided by Polya [11], he uses Fatou’s lemma by considering the generating function of the trace operator (which is a rational function). Alternative proofs were given by H. Lenstra and P. Ponomarev [4] independently using complementary modules.

The theorem of Polya doesn’t hold when we consider an infinite subset of natural numbers. For example, for the algebraic number $\alpha=1/\sqrt{2}$, we have $\text{Tr}_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(1/\sqrt{2})^n$ is always an integer whenever $n \equiv 1 \pmod{2}$. Hence we need to assume ‘non-zero integer’ condition when we consider the Polya’s question for an infinite subset of the natural numbers. In [2], P. Corvaja and U. Zannier, using the subspace theorem, proved the following. *Suppose α is an algebraic number and let E be an infinite subset of \mathbb{N} . For each $n \in E$, suppose there exists a positive integer q_n such that $\lim_{n \in E} (\log q_n)/n = 0$ and $\text{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(q_n \alpha^n) \in \mathbb{Z}$. Then α is either the h -th root of a rational number for some positive integer h or an algebraic integer.* P. Philippon

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and P. Rath [9] proved a similar result by replacing the integer q_n with a constant which is a non-zero algebraic number.

A finite version of Polya’s question was refined further by B. de Smit [4] who explicitly computed the constant C in terms of α : *If $\text{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^m) \in \mathbb{Z}$ for $1 \leq m \leq C$, then α is an algebraic integer.* A finite constant is expected because we need to evaluate only finitely many elementary symmetric functions at the Galois conjugates of α to determine whether the given value α is an algebraic integer. The constant C in B. de Smit’s result is optimal.

In this paper, we study two problems, namely,

- (1) We consider the generalised power sums of the form $\lambda_1\alpha_1^\ell + \dots + \lambda_k\alpha_k^\ell$ where λ_i, α_j are algebraic numbers and $\ell \geq 1$ is any integer. If the power sum is an algebraic integer for infinitely many ℓ ’s, then under what conditions, we can conclude α_j ’s are algebraic integers? As applications of this result (Theorem 2.1), we consider the analogous situation over polynomials, group rings, function fields and a linear combination of trace powers of algebraic numbers.
- (2) Some special cases wherein we can restrict ℓ to a finite (effective) set for a generalised power sum.

The main ideas of our work are coherent with those of [2], [4] and [9].

2. OUR RESULTS

Given a set of non-zero algebraic numbers $\alpha_1, \dots, \alpha_k$, we partition them into equivalence classes by the following equivalence relation:

$$(2.1) \quad \alpha_i \sim \alpha_j \text{ if and only if their ratio is a root of unity.}$$

The algebraic numbers $\alpha_1, \dots, \alpha_k$ are said to be *non-degenerate* if they have k -equivalence classes. In general, by the equivalence relation in (2.1), a tuple $(\alpha_1, \dots, \alpha_k)$ induces a partition on the index set $\mathcal{I} = \{1, \dots, k\}$, that is, $\mathcal{I} = \cup_j I_j$, where for each j , the set $\{\alpha_r : r \in I_j\}$ is an equivalence class under (2.1).

We denote the field of all algebraic numbers by $\overline{\mathbb{Q}}$ and the ring of all algebraic integers by $\overline{\mathbb{Z}}$, and we set $\zeta_h := e^{2\pi i/h}$ for an integer $h \geq 2$. Now we state one of the main theorems as follows.

Theorem 2.1. *Let $\mathcal{L}(X_1, \dots, X_k) = \sum_{i=1}^k \lambda_i X_i$ be a linear form with coefficients λ_i in $\overline{\mathbb{Q}}^\times$. Let $(\alpha_1, \dots, \alpha_k) \in \overline{\mathbb{Q}}^{\times k}$ be a given k -tuple of algebraic numbers such that*

$$\mathcal{L}(\alpha_1^n, \dots, \alpha_k^n) \in \overline{\mathbb{Z}},$$

for n in an infinite set $\mathfrak{S} \subset \mathbb{N}$. Then for each subset I_j of $\mathcal{I} = \{1, \dots, k\}$ corresponding to an equivalence class induced by (2.1), one of the following holds true:

- (1) *We have $\sum_{a \in I_j} \lambda_a \alpha_a^n = 0$ for all but finitely many values of $n \in \mathfrak{S}$.*
- (2) *The numbers α_i are algebraic integers for all $i \in I_j$.*

Now we proceed to provide some consequences of Theorem 2.1. Theorem 2.2 and Theorem 2.3 follow immediately from Theorem 2.1, and the conditions imposed on these theorems are necessary to ensure that a given algebraic number is an algebraic integer.

2.1. Polynomial values. Here the linear form $\mathcal{L}(X_1, \dots, X_k)$ can be replaced with a polynomial $P(X_1, \dots, X_k)$ satisfying some mild hypotheses.

Theorem 2.2. *Let $P(X_1, \dots, X_k)$ be a polynomial with algebraic coefficients and assume that $P(X_1, 0, \dots, 0, 0)$ is a non-constant polynomial. Let $\alpha_1, \dots, \alpha_k$ be multiplicatively independent non-zero algebraic numbers such that $P(\alpha_1^n, \dots, \alpha_k^n) \in \overline{\mathbb{Z}}$ for infinitely many positive integers n . Then the number α_1 is an algebraic integer.*

Remark 2.1. Theorem 2.2 is no longer true if we remove the condition on $P(X_1, 0, \dots, 0)$ is a non-constant polynomial. For instance, let $P(X, Y) = XY + 1$ and fix a prime $p \neq 2$. Choose $\alpha_1 = 1/p$ and $\alpha_2 = 2p$ and note that $P(\alpha_1^n, \alpha_2^n) = 2^n + 1 \in \mathbb{Z}$ for all natural numbers n .

2.2. Group rings. Let α be a non-zero algebraic number, K_α be the Galois closure of $\mathbb{Q}(\alpha)$ and $G_\alpha = \text{Gal}(K_\alpha/\mathbb{Q})$ be the Galois group. We identify the index set \mathcal{I} of Theorem 2.1 as the elements of G_α , for convenience, and consider the group ring $\overline{\mathbb{Q}}[G_\alpha]$.

Theorem 2.3. *Let $f \in \overline{\mathbb{Q}}[G_\alpha]$ be a non-zero element such that $f(\alpha^n) \in \overline{\mathbb{Z}} \setminus \{0\}$ for n in an infinite subset $\mathfrak{S} \subset \mathbb{N}$. Then α is an algebraic integer.*

Remark 2.2. In particular, Theorem 2.3 proves the following: *If $\sigma(\alpha^n) - \alpha^n$ is a non-zero algebraic integer for infinitely many n , then α is an algebraic integer.* However, the same conclusion is not possible by applying the trace operator because $\text{Tr}_{K/\mathbb{Q}}(\sigma(\alpha^n) - \alpha^n) = 0$. One notes that this fact can be obtained by applying the number field version of Ridout's theorem, see for instance, Corollary 1.2 in Chapter 7 of S. Lang [7] (or Theorem D.2.1 in [5]).

2.3. Action under the trace map. Let α be a non-zero algebraic number of degree d and let $\alpha = \alpha_1, \dots, \alpha_d$ be all the Galois conjugates of α . If $P(X) \in \mathbb{Q}[X]$ is a non-zero polynomial of α , then

$$\text{Tr}_{K/\mathbb{Q}}(P(\alpha^n)) = Q(\alpha_1^n, \dots, \alpha_d^n),$$

where $Q(X_1, \dots, X_d) = P(X_1) + \dots + P(X_d)$. Though the polynomial Q satisfies the hypothesis of Theorem 2.2, the numbers $\alpha_1, \alpha_2, \dots, \alpha_d$ need not be multiplicatively independent. However, we have Theorem 2.4 for the trace operator.

Theorem 2.4. *Let α be a non-zero algebraic number and let $P(X) = \lambda_D X^D + \dots + \lambda_0 \in \overline{\mathbb{Q}}[X]$ be a non-constant polynomial of degree D and let $L = \mathbb{Q}(\lambda_0, \dots, \lambda_D, \alpha)$. If $\text{Tr}_{L/\mathbb{Q}}(P(\alpha^n)) \in \mathbb{Z}$ for each n in an infinite set \mathfrak{S} of natural numbers, then either α is an algebraic integer or for each $i = 1, 2, \dots, D$, we have $\text{Tr}_{L/\mathbb{Q}}(\lambda_i \alpha^{in}) = 0$ for all but finitely many $n \in \mathfrak{S}$.*

When we consider a multivariable generalisation of Theorem 2.4, we may have that the trace operator vanishes for a subsum for trivial reasons. For simplicity, we consider a linear form in several variables.

Theorem 2.5. *Suppose $\alpha_1, \dots, \alpha_k, \lambda_1, \dots, \lambda_k$ be distinct non-zero algebraic numbers. Let $L = \mathbb{Q}(\alpha_i, \lambda_i \mid 1 \leq i \leq k)$, K be its Galois closure and h be the order of the torsion subgroup of K^\times over \mathbb{Q} . Suppose $\text{Tr}_{L/\mathbb{Q}}(\lambda_1 \alpha_1^n + \dots + \lambda_k \alpha_k^n) \in \mathbb{Z}$ for n in an infinite subset $\mathfrak{S} \subset \mathbb{N}$. If α_1 is not an algebraic integer, then there exists an integer $a \in \{0, \dots, h-1\}$ and a subset I of $\{1, \dots, k\}$ containing 1 such that*

$$\text{Tr}_{L/\mathbb{Q}} \left(\sum_{i \in I} \lambda_i \alpha_i^a \right) = 0,$$

where for each $i \in I$, there exists $\sigma_i \in \text{Gal}(K/\mathbb{Q})$, the Galois group of K , such that $\sigma_i(\alpha_i)/\alpha_1$ is a root of unity.

We have the following interesting corollary as a consequence of this result.

Corollary 2.1. *Let $\alpha_1, \dots, \alpha_k, \lambda_1, \dots, \lambda_k$ be non-zero algebraic numbers. Let $L = \mathbb{Q}(\alpha_i, \lambda_i \mid 1 \leq i \leq k)$, K be its Galois closure and h be the order of the torsion subgroup of the multiplicative group K^\times . Suppose that no subsum of $\lambda_1\alpha_1^a + \dots + \lambda_k\alpha_k^a$ vanishes, under the trace map, for each $a \in \{0, 1, \dots, h-1\}$. If $\text{Tr}_{L/\mathbb{Q}}(\lambda_1\alpha_1^n + \dots + \lambda_k\alpha_k^n) \in \mathbb{Z}$ for infinitely many natural numbers n , then each α_i is an algebraic integer for all $i = 1, \dots, k$.*

Corollary 2.1 can be considered as a multidimensional generalisation of a result of P. Philippon and P. Rath in [9].

2.4. Determining the nature of rational functions. We consider a function field (of characteristic 0) analogue of Theorem 2.1 in the simplest setting after imposing some additional restrictions.

Theorem 2.6. *Let $f_1(X), \dots, f_k(X)$ be non-constant rational functions with algebraic coefficients and $\lambda_1, \dots, \lambda_k$ be non-zero algebraic numbers. Assume that the ratio $f_i(X)/f_j(X)$ is not a constant function for each $i \neq j$. If*

$$(2.2) \quad \sum_{i=1}^k \lambda_i (f_i(X))^n \in \overline{\mathbb{Z}}[X]$$

for n in an infinite subset $\mathfrak{S} \subset \mathbb{N}$, then each $f_i(X) \in \overline{\mathbb{Z}}[X]$.

It might be possible to show each of the functions $f_i(X)$ are polynomial functions without appealing to the subspace theorem. However, the approach here is to deduce the nature of rational function via its specialisations.

2.5. Determining algebraic integers in finite iteration. In the work of B. de Smit [4], a finite bound on ℓ in $\text{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^\ell)$ was given in order to determine whether α is an algebraic integer. We provide the following generalisations:

Theorem 2.7. *Let α_1 be a non-zero algebraic number and $\alpha_2, \dots, \alpha_k$ be all the other Galois conjugates of α_1 for some integer $k \geq 2$. Let K be the Galois closure of $\mathbb{Q}(\alpha_1)$ and $[K : \mathbb{Q}] = d \geq k$. For any integer b_1, \dots, b_k (not necessarily distinct) such that $b_1 + \dots + b_k = n \neq 0$, if $\text{Tr}_{K/\mathbb{Q}}(b_1\alpha_1^j + \dots + b_k\alpha_k^j) \in \mathbb{Z}$ for all $j = 1, 2, \dots, d + d[\log_2(nd)] + 1$, then α_1 is an algebraic integer and so is α_j for each $j \geq 2$.*

We now look at a multidimensional analogue of B. de Smit's result more generally, not just for trace operators, but for a given linear recurrence sequence. Let $\alpha_i \in \overline{\mathbb{Q}}^\times$ be distinct algebraic numbers, $\lambda_i \in \overline{\mathbb{Q}}^\times$ for all $1 \leq i \leq k$, $K = \mathbb{Q}(\alpha_1, \dots, \alpha_k, \lambda_1, \dots, \lambda_k)$ and \mathcal{O}_K be its ring of integers. We ask Question 1:

Question 1. Does there exist a constant C (depending on α_i and λ_i) such that

$$\lambda_1\alpha_1^\ell + \dots + \lambda_k\alpha_k^\ell \in \mathcal{O}_K \text{ for all } 1 \leq \ell \leq C \implies \alpha_j \in \mathcal{O}_K \text{ for all } 1 \leq j \leq k?$$

When $k = 1$, one can answer Question 1 positively with $C = 1 + \max_{\mathfrak{p}} |v_{\mathfrak{p}}(\lambda_1)|$ where \mathfrak{p} runs through the non-zero prime ideals in \mathcal{O}_K and $v_{\mathfrak{p}}$ denotes the valuation at \mathfrak{p} . Here, we answer this question when $k = 2$ and highlight some difficulties in generalising this proof for arbitrary k in Remark 5.1.

Theorem 2.8. *Let $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ be given non-zero algebraic numbers. Let $K = \mathbb{Q}(\lambda_1, \lambda_2, \alpha_1, \alpha_2)$ and let*

$$C = 2 + \left[\frac{1}{2} \max_{\mathfrak{P}} |v_{\mathfrak{P}}(\lambda_1 \lambda_2 (\alpha_1 - \alpha_2)^2)| \right] + \max_{\mathfrak{P}} |v_{\mathfrak{P}}(\lambda_1 \lambda_2)|,$$

where \mathfrak{P} runs through the non-zero prime ideals in \mathcal{O}_K . If

$$\lambda_1 \alpha_1^\ell + \lambda_2 \alpha_2^\ell \in \mathcal{O}_K \text{ for all } 1 \leq \ell \leq C,$$

then $\alpha_1, \alpha_2 \in \mathcal{O}_K$.

3. PRELIMINARIES

In this section, we state the propositions/theorems required for the proof of our theorems. We also need some results which are applications of the Schmidt Subspace Theorem, formulated by Evertse and Schlickewei. For a reference, see [1, Chapter 7], [12, Chapter V, Theorem 1D'], and [13, Page 16, Theorem II.2].

Let K be a number field which is a Galois extension over \mathbb{Q} . Let M_K be the set of all places on K and M_∞ be the set of all archimedean places on K . For each place $w \in M_K$, let K_w denote the completion of the number field K with respect to w and $d(w) = [K_w : \mathbb{Q}_v]$, where v is the restriction of w to \mathbb{Q} . For every $w \in M_K$ whose restriction on \mathbb{Q} is v and $\alpha \in K$, we define the normalized absolute value $|\cdot|_w$ as follows:

$$(3.1) \quad |\alpha|_w := |\text{Norm}_{K_w/\mathbb{Q}_v}(\alpha)|_v^{\frac{1}{[K:\mathbb{Q}]}}.$$

Indeed if $w \in M_\infty$, then there exists an automorphism $\sigma \in \text{Gal}(K/\mathbb{Q})$ of K such that for all $x \in K$,

$$|x|_w = |\sigma(x)|^{d(K)/[K:\mathbb{Q}]},$$

where $d(K) = 1$ if $K \subset \mathbb{R}$ and $d(K) = 2$ otherwise. Note that since K is Galois over \mathbb{Q} , the embeddings are either totally real or totally complex, and hence the function $d(K)$ is constant. Thus, under the definition (3.1), the product formula

$\prod_{\omega \in M_K} |x|_\omega = 1$ holds true for any $x \in K^\times$. For a non-zero vector $\mathbf{x} = (x_1, \dots, x_n) \in K^n$, the projective height, $H(\mathbf{x})$, is defined by

$$H(\mathbf{x}) = \prod_{\omega \in M_K} \max\{|x_1|_\omega, \dots, |x_n|_\omega\}.$$

We require the following results from [6].

Proposition 3.1 (A. Kulkarni, N. Mavraki and K. D. Nguyen [6]). *Let $\alpha_1, \dots, \alpha_k$ be non-degenerate non-zero algebraic numbers and let $\lambda_1, \dots, \lambda_k$ be non-zero algebraic numbers. Then there are at most finitely many natural numbers n satisfying*

$$\lambda_1 \alpha_1^n + \dots + \lambda_k \alpha_k^n = 0.$$

Theorem 3.1 (A. Kulkarni, N. Mavraki and K. D. Nguyen [6]). *Let K be a number field which is Galois over \mathbb{Q} and S be a finite set of places, containing all the archimedean places. Let $\lambda_1, \dots, \lambda_k$ be non-zero elements of K . Let $\varepsilon > 0$ be a*

positive real number and $\omega \in S$ be a distinguished place. Let \mathfrak{E} be the set of all $(u_1, \dots, u_k) \in (\mathcal{O}_S^\times)^k$ which satisfy the inequality

$$(3.2) \quad 0 < \left| \sum_{j=1}^k \lambda_j u_j \right|_\omega \leq \frac{\max\{|u_1|_\omega, \dots, |u_k|_\omega\}}{H(u_1, \dots, u_k, 1)^\varepsilon},$$

where \mathcal{O}_S^\times is the ring of S -units in K . If \mathfrak{E} is an infinite set, then there exist $c_1, \dots, c_k \in K$, not all zero, such that

$$c_1 u_1 + \dots + c_k u_k = 0$$

holds true for infinitely many elements (u_1, \dots, u_k) of \mathfrak{E} .

We prove Proposition 3.2 regarding the elements of Galois group fixing the equivalence class of a non-zero algebraic element α (given by (2.1)). This is of independent interest.

Proposition 3.2. *Let α be a non-zero algebraic number, and K be any Galois extension of \mathbb{Q} containing α with its Galois group $G = \text{Gal}(K/\mathbb{Q})$. Let*

$$H = \{\sigma \in G : \sigma(\alpha) \sim \alpha\}$$

be a subset of G . Then the following statements are true;

- (1) H is a subgroup of G .
- (2) For any given $\tau \in G$, we have $\{\sigma \in G \mid \sigma(\tau(\alpha)) \sim \tau(\alpha)\} = \tau H \tau^{-1}$.

Proof. If $\sigma_a \in H$ then we have $\sigma_a(\alpha) = \zeta_h^{w_a} \alpha$ for some integer w_a . Since G permutes the roots of unity, so does H . This shows that H is closed under inversion and composition of automorphisms in G . Hence H is a subgroup of G . This proves (1).

Let $\sigma \in H$ be any element. Then we have

$$\sigma(\alpha) \sim \alpha \Leftrightarrow \tau(\sigma(\alpha)) \sim \tau(\alpha) \Leftrightarrow (\tau\sigma\tau^{-1})(\tau\alpha) \sim \tau(\alpha),$$

proving the second statement. □

We shall state the following basic and well-known lemma which roughly says ‘integrality’ is a local phenomenon.

Lemma 3.1. *Let $\alpha \in \bar{\mathbb{Q}}$ be an algebraic number. Then α is an algebraic integer if and only if α is integral over \mathbb{Z}_p for every prime number p where \mathbb{Z}_p is the ring of p -adic integers.*

Lemma 3.2 is also basic and well-known and hence we omit the proof here.

Lemma 3.2. *Let K be a finite extension over \mathbb{Q}_p of degree d where \mathbb{Q}_p is the field of p -adic numbers. Let $\alpha \in K$ be an element such that $\alpha \notin \mathcal{O}_K$, the local ring of K . Then $\beta := \alpha^{-1} \in \mathcal{O}_K$. Furthermore, if the characteristic polynomial of β is*

$$f_{K|\mathbb{Q}_p}(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0 \in \mathbb{Z}_p[x],$$

then $a_d = 1$ and $a_i \in p\mathbb{Z}_p$ for all $i = 0, 1, \dots, d-1$ where $p\mathbb{Z}_p$ is the unique maximal ideal of \mathbb{Z}_p .

We need Lemma 3.3 in the proof of Theorem 2.7.

Lemma 3.3. *Let p be a given prime number and let K be a finite Galois extension over \mathbb{Q}_p of degree d . Let $\alpha \in K$ be an element such that $\alpha \notin \mathcal{O}_K$ and β be a Galois conjugate of α . Let b_1 and b_2 be given p -adic integers such that $b_1 + b_2 \neq 0$. Then for each integer $\ell \geq 0$, there exists an integer $i \in \{1, 2, \dots, d\}$ such that $b_1\alpha^{\ell+i} + b_2\beta^{\ell+i} \neq 0$.*

Proof. Since $\alpha \notin \mathcal{O}_K$, by Lemma 3.2, it is clear that $\alpha^{-1} \in \mathcal{O}_K$ and let $f(x) = x^d + a_{d-1}x^{d-1} + \dots + a_1x + a_0 \in \mathbb{Z}_p[x]$ be the characteristic polynomial of α^{-1} . Since β is a Galois conjugate of α , it is clear that $f(\alpha^{-1}) = 0 = f(\beta^{-1})$. We see that for every integer $m \geq 0$, we have

$$\alpha^m = -a_{d-1}\alpha^{m+1} - \dots - a_0\alpha^{m+d} \text{ and } \beta^m = -a_{d-1}\beta^{m+1} - \dots - a_0\beta^{m+d}.$$

Therefore, we get

$$(3.3) \quad b_1\alpha^m + b_2\beta^m = -a_{d-1}(b_1\alpha^{m+1} + b_2\beta^{m+1}) - \dots - a_0(b_1\alpha^{m+d} + b_2\beta^{m+d}).$$

Now we prove the assertion by induction on ℓ .

We put $m = 0$ in (3.3), we get

$$0 \neq b_1 + b_2 = -a_{d-1}(b_1\alpha^1 + b_2\beta^1) - \dots - a_0(b_1\alpha^d + b_2\beta^d)$$

which implies the assertion when $\ell = 0$. Assume the assertion is true for some integer $\ell > 0$. That is, there exists an integer $i \in \{1, 2, \dots, d\}$ such that $b_1\alpha^{\ell+i} + b_2\beta^{\ell+i} \neq 0$. If $i \geq 2$, then $\ell + i = \ell + 1 + j$ for some integer $j \in \{1, 2, \dots, d\}$ and then we are done. Hence we assume that $i = 1$. That is, $b_1\alpha^{\ell+1} + b_2\beta^{\ell+1} \neq 0$. Now, put $m = \ell + 1$ in (3.3) to get the assertion in this case as well. Hence the lemma. □

We conclude this section by discussing a proof of Fatou’s Lemma. This was proved by Pisot [10] for number fields and the proof was adapted based on Fatou’s work [3]. We provide a slightly different proof along the lines mentioned in B. de Smit [4]. Before proceeding further, for a non-zero algebraic number α , we define the denominator of α (denoted by $\text{den}(\alpha)$) to be the smallest positive integer n such that $n\alpha$ is an algebraic integer.

Proposition 3.3 (Fatou’s lemma). *Let K be a number field and $f(X) \in K(X) \cap \mathcal{O}_K[[X]]$ such that $f(X) = g(X)/h(X)$ where $g(X), h(X) \in K[X]$ are coprime in $K[X]$ and $h(0) = 1$. Then $g(X), h(X) \in \mathcal{O}_K[X]$.*

Proof. Given that $f(X) = g(X)/h(X)$ where $g(X), h(X) \in K[x]$ are coprime and $h(0) = 1$. Then there exist two polynomials $r(X), s(X) \in \mathcal{O}_K[X]$ such that $r(X)g(X) + s(X)h(X) = c$ for some non-zero constant $c \in \mathcal{O}_K$. Therefore, we have

$$(3.4) \quad \frac{c}{h(X)} \in \mathcal{O}_K[[X]].$$

Now let $h(X) = \prod_{i=1}^k (1 - a_i X)^{c_i}$ in $\overline{\mathbb{Q}}[X]$ for some integers $c_i \geq 1$ and some distinct $a_i \in \overline{\mathbb{Q}}$. For each i with $1 \leq i \leq k$, multiplying (3.4) by the factor

$$\left[\prod_{\substack{j=1 \\ j \neq i}}^k \text{den}(a_j)^{c_j} (1 - a_j X)^{c_j} \right] \left[\text{den}(a_i)^{c_i-1} (1 - a_i X)^{c_i-1} \right],$$

we define a new formal power series $\tilde{f}_i(X) := c'(1 - a_i X)^{-1}$ for some c' depending on $\text{den}(a_j)$ and c_j for all $1 \leq j \leq k$. Since we are multiplying by a polynomial whose coefficients are algebraic integers, we get $\tilde{f}_i(X) \in \mathcal{O}_L[[X]]$ for some finite extension L over K in which $h(X)$ splits completely. Therefore for any non-zero prime ideal \mathfrak{P} in \mathcal{O}_L , we have

$$v_{\mathfrak{P}}(c' a_i^n) \geq 0 \text{ for all } n \implies v_{\mathfrak{P}}(a_i) \geq 0$$

and consequently, we get a_i is an algebraic integer. Therefore $h(X) \in K[X] \cap \mathcal{O}_L[X] = \mathcal{O}_K[X]$ and hence $g(X) = f(X)h(X) \in \mathcal{O}_K[X]$ as both $f(X), h(X)$ take values in \mathcal{O}_K . □

4. PROOFS OF THEOREMS 2.1 TO 2.6

Proof of Theorem 2.1. Let $K = \mathbb{Q}(\alpha_1, \dots, \alpha_k)$ be a number field and h be the order of the torsion subgroup of K^\times . We partition $\{\alpha_1, \dots, \alpha_k\} = \bigcup_{l=1}^d \{\alpha_a : a \in I_l\}$ by the equivalence relation in (2.1). For each index set I_l , let β_l be a representative of the corresponding equivalence class $\{\alpha_a : a \in I_l\}$, and for $a \in I_l$, let $\alpha_a = \beta_l \zeta_h^{\omega_{a,l}}$ for some integer $\omega_{a,l}$. The numbers β_1, \dots, β_d are representatives (elements) of disjoint equivalence classes and hence the tuple $(\beta_1, \dots, \beta_d)$ is non-degenerate.

Let l be a fixed natural number with $1 \leq l \leq d$ and assume that assertion (1) is not true for I_l . Then, for infinitely many $n \in \mathfrak{S}$, the sum $\sum_{a \in I_l} \lambda_a \alpha_a^n \neq 0$. In particular, there exists an infinite subset $\mathfrak{S}_m := \{n \in \mathfrak{S} : n \equiv m \pmod{h}\}$ for some $m \in \{0, 1, \dots, h - 1\}$ such that $\sum_{a \in I_l} \lambda_a \alpha_a^n \neq 0$ for each $n \in \mathfrak{S}_m$. Therefore, for each $n \in \mathfrak{S}_m$, we have

$$\begin{aligned} \mathcal{L}(\alpha_1^n, \dots, \alpha_k^n) &= \sum_{i=1}^k \lambda_i \alpha_i^n \\ &= \sum_{r=1}^d \sum_{a \in I_r} \lambda_a \alpha_a^n \\ &= \sum_{r=1}^d \sum_{a \in I_r} \lambda_a \zeta_h^{n\omega_{a,r}} \beta_r^n = \sum_{r=1}^d \kappa_{r,m} \beta_r^n =: \mathcal{L}_m(\beta_1^n, \dots, \beta_d^n), \end{aligned}$$

where $\kappa_{r,m} := \sum_{a \in I_r} \lambda_a \zeta_h^{m\omega_{a,r}}$, as $n \equiv m \pmod{h}$. Since $\kappa_{l,m} \beta_l^n = \sum_{a \in I_l} \lambda_a \alpha_a^n$, we have $\kappa_{l,m} \neq 0$. Furthermore, since $(\beta_1, \dots, \beta_d)$ is a non-degenerate tuple, by Proposition 3.1, $\mathcal{L}(\beta_1^n, \dots, \beta_d^n) \in \overline{\mathbb{Z}} \setminus \{0\}$ for all but finitely many values of $n \in \mathfrak{S}_m$. We remove the finite numbers $n \in \mathfrak{S}_m$ for which $\mathcal{L}(\beta_1^n, \dots, \beta_d^n) = 0$.

We proceed to prove that β_l is an algebraic integer via contradiction.

Let S be a suitable finite subset of M_K containing all the archimedean places such that β_j is an S -unit for each $j = 1, 2, \dots, d$. Assume that β_l is not an algebraic integer. Then there exists a finite place $\omega \in S$ such that $|\beta_l|_\omega > 1$. Choose $\varepsilon > 0$ such that $\varepsilon < \frac{\log |\beta_l|_\omega}{\log H(\beta_1, \dots, \beta_d, 1)}$. For all $n \in \mathfrak{S}_m$, we have $|\beta_l|_\omega^n H(\beta_1^n, \dots, \beta_d^n, 1)^{-\varepsilon} > 1$

because by the choice of ε and the height inequality

$$H(x_1y_1 : \dots, x_dy_d : 1) \leq H(x_1 : \dots, x_d : 1)H(y_1 : \dots, y_d : 1).$$

Since \mathfrak{S}_m is an infinite set, Theorem 3.1 (applied to the linear form \mathcal{L}_m consisting of the subtuple (\dots, β_l, \dots) for which $\kappa_{l,m} \neq 0$) asserts that

$$\sum_{i=1}^d b_i \beta_i^n = 0$$

holds true for infinitely many natural numbers $n \in \mathfrak{S}_m$, where $b_i \in K$ and not all zero. This is a contradiction to Proposition 3.1. Therefore β_l is an algebraic integer. \square

Proof of Theorem 2.2. Given polynomial $P(X_1, \dots, X_k)$, we write a typical monomial in P as $\prod_{j=1}^k X_j^{b_j} = X_{\mathbf{i}}$ where $\mathbf{i} = (b_1, \dots, b_k)$. Also, we write the coefficient of the monomial $X_{\mathbf{i}}$ in P as $a_{\mathbf{i}}$. Let

$$\mathcal{I} = \{(b_1, \dots, b_k) : \mathbf{i} = (b_1, \dots, b_k) \text{ and } X_{\mathbf{i}} \text{ appears in } P\}$$

be the index set.

Now, consider the linear form $\mathcal{L}((Y_{\mathbf{i}})_{\mathbf{i} \in \mathcal{I}}) = \sum_{\mathbf{i} \in \mathcal{I}} a_{\mathbf{i}} Y_{\mathbf{i}}$. Therefore, by setting $\alpha_{\mathbf{i}} :=$

$$\prod_{j=1}^k \alpha_j^{b_j} \text{ for each } \mathbf{i} = (b_1, \dots, b_k) \in \mathcal{I}, \text{ we see that } P(\alpha_1, \dots, \alpha_k) = \mathcal{L}((\alpha_{\mathbf{i}})_{\mathbf{i} \in \mathcal{I}}).$$

By hypothesis, $\alpha_1, \dots, \alpha_k$ are multiplicatively independent. Therefore the sequence $(\alpha_1^n, \dots, \alpha_k^n)$ is Zariski-dense on the k -dimensional space \mathbb{A}^k ; same holds for every infinite subsequence. In particular, $P(\alpha_1^n, \dots, \alpha_k^n)$ cannot vanish infinitely often. Thus, it follows that $P(\alpha_1^n, \dots, \alpha_k^n) \in \mathbb{Z} \setminus \{0\}$ for infinitely many natural number n (by hypothesis).

By hypothesis, since $P(X_1, 0, \dots, 0)$ is not a constant polynomial, we let $P(X_1, 0, \dots, 0) = \sum_{i=0}^d c_i X_1^i$, with $c_d \neq 0$ and therefore $(d, 0, \dots, 0) \in \mathcal{I}$. Let I_1 be an equivalence class induced by (2.1) corresponding to a subset, say, \mathcal{I}_1 of \mathcal{I} such that $(d, 0, \dots, 0) \in \mathcal{I}_1$ (or equivalently, $\alpha_1^d \in I_1$). Since $\alpha_1, \dots, \alpha_k$ are multiplicatively independent, we get $\mathcal{I}_1 \subseteq \{(c, 0, \dots, 0) \mid 0 \leq c \leq d\}$. Then we have the following cases to consider:

- (1) If $(c, 0, \dots, 0) \in \mathcal{I}_1$ for some non-negative integer $c < d$, then $\alpha_1^d \sim \alpha_1^c$, and therefore α_1^{d-c} is a root of unity. This implies that α_1 is a root of unity.
- (2) If $\mathcal{I}_1 = \{(d, 0, \dots, 0)\}$, then by Theorem 2.1, α_1^d is an algebraic integer. Therefore, α_1 is an algebraic integer.

In both cases, we are done. \square

Proof of Theorem 2.3. Let $f = \sum_{\sigma \in G_\alpha} \lambda_\sigma \sigma$ be a non-zero element of $\overline{\mathbb{Q}}[G_\alpha]$. Then

we have

$$f(\alpha^n) = \sum_{\sigma \in G_\alpha} \lambda_\sigma \sigma(\alpha^n) = \sum_{\sigma \in G_\alpha} \lambda_\sigma \sigma(\alpha)^n.$$

Therefore, by letting $\overline{X} = (X_\sigma)_{\sigma \in G_\alpha}$, the linear form $\mathcal{L}(\overline{X}) = \sum_{\sigma \in G_\alpha} \lambda_\sigma X_\sigma$ and $\overline{\alpha} := (\sigma(\alpha))_{\sigma \in G_\alpha}$, we note that $\mathcal{L}(\overline{\alpha}^n) = f(\alpha^n)$. Hence by hypothesis, we have $\mathcal{L}(\overline{\alpha}^n) \in \overline{\mathbb{Z}} \setminus \{0\}$ for all $n \in \mathfrak{S}$.

The equivalence relation in (2.1), induces a partition on the index set G_α , that is, $G_\alpha = \cup_{j=1}^d I_j$. If possible, we suppose α (and therefore all the Galois conjugates of α) is not an algebraic integer. By Theorem 2.1, for every equivalence class I_j we have

$$\sum_{\sigma \in I_j} \lambda_\sigma \sigma(\alpha)^n = 0 \text{ for all but finitely many } n \in \mathfrak{S}.$$

Therefore, for all but finitely many values of $n \in \mathfrak{S}$, we have

$$f(\alpha^n) = \sum_{j=1}^d \sum_{\sigma \in I_j} \lambda_\sigma \sigma(\alpha)^n = 0,$$

a contradiction as $f(\alpha^n) \in \overline{\mathbb{Z}} \setminus \{0\}$ for all $n \in \mathfrak{S}$ and \mathfrak{S} is an infinite set. □

For the proof of Theorems 2.4 and 2.5, the index set, $\mathcal{I} \times G$, will be a finite subset of $\mathbb{N} \times \text{Gal}(K/\mathbb{Q})$ for an appropriate Galois extension K/\mathbb{Q} . Then we set the linear form as

$$\mathcal{L}(\overline{X}) := \mathcal{L}((X_{i,\sigma})_{(i,\sigma) \in \mathcal{I} \times G}) = \sum_{(i,\sigma) \in \mathcal{I} \times G} \sigma(\lambda_i) X_{i,\sigma}.$$

Proof of Theorem 2.4. First, we can assume that α is not a root of unity (for otherwise, we are done). Since $\text{Tr}_{L/\mathbb{Q}} = \text{Tr}_{F/\mathbb{Q}} \text{Tr}_{L/F}$ for an intermediate field F of L , we further reduce our computations to the field $F = \mathbb{Q}(\alpha)$. That is, we can assume that $\lambda_i \in F$ for each i .

Let K be the Galois closure of F and its Galois group $G = \text{Gal}(K/\mathbb{Q})$. We set the index set $\mathcal{I} \times G = \{(k, \sigma) \mid \lambda_k \neq 0, 1 \leq k \leq D, \sigma \in G\}$. For every algebraic number γ , we define the tuple $\overline{\gamma} := (\sigma(\gamma)^k)_{(k,\sigma) \in \mathcal{I} \times G}$. Therefore, by hypothesis, we have $\mathcal{L}(\overline{\alpha}^n) = \text{Tr}_{F/\mathbb{Q}}(P(\alpha^n)) = \sum_{i=0}^D \text{Tr}_{F/\mathbb{Q}}(\lambda_i \alpha^{in}) \in \mathbb{Z}$ for each n in an infinite subset $\mathfrak{S} \subseteq \mathbb{N}$.

If α is not an algebraic integer (and so are the conjugates of α and their powers), then by expanding the trace operator and by Theorem 2.1, we get for every equivalence class and the corresponding index set $I_j \times H \subset \mathcal{I} \times G$ for some subset $H \subset G$ such that

$$(4.1) \quad \sum_{i \in I_j} b_i \alpha_i^n = 0,$$

for all but finitely many n where b_i denotes the conjugates of λ_l for some finite collection and α_i denotes some conjugate of α^l for some $l \leq D$ appropriately.

Note that for any $(i, \sigma), (k, \tau) \in I_j \times H$, we see that $\sigma(\alpha^i) \sim \tau(\alpha^k)$ which implies that $\alpha^i \sim \delta(\alpha^k)$ for some $\delta \in G$. Now, we claim that *if $\alpha^l \sim \sigma(\alpha)^k$ for some $1 \leq l, k \leq D$ and for some $\sigma \in G$, then $l = k$* . In order to prove this claim, we use the properties of logarithmic Weil height $h(x) = \log |H(x)|$. As l and k are positive integers, since

$$lh(\alpha) = h(\alpha^l) = h(\zeta \sigma(\alpha)^k) = h(\alpha^k) = kh(\alpha)$$

(where ζ is a root of unity), we get $l = k$ as $h(\alpha) > 0$, which proves the claim.

Suppose $\lambda_l \alpha^l$ is one of the terms in the above summand. Then by the claim, (4.1) becomes

$$\sum_{\sigma \in H} \sigma(\lambda_l \alpha^{ln}) = 0$$

for all but finitely many n . Applying the trace operator on both sides, we get $|H| \text{Tr}_{F/\mathbb{Q}}(\lambda_l \alpha^{ln}) = 0$ for infinitely many n . Since $|H| \geq 1$, we get the assertion. \square

Proof of Theorem 2.5. Let K be the Galois closure of L and its Galois group $G = \text{Gal}(K/\mathbb{Q})$. We write the index set $\mathcal{I} \times G = \{(i, \sigma) \mid 1 \leq i \leq k, \sigma \in G\}$. For the tuple $\overline{\alpha}^n := (\sigma(\alpha_i)^n)_{\sigma \in G, 1 \leq i \leq k}$, we note that $\mathcal{L}(\overline{\alpha}^n) = \text{Tr}_{L/\mathbb{Q}}(\lambda_1 \alpha_1^n + \dots + \lambda_k \alpha_k^n) \in \mathbb{Z}$ for n in an infinite subset \mathfrak{S} of \mathbb{N} .

Since α_1 is not an algebraic integer, by hypothesis, by applying Theorem 2.1 to the linear form $\mathcal{L}(\overline{\alpha}^n)$, there exists an equivalence class and the corresponding index set I_1 containing $(1, \sigma_1)$ (σ_1 denotes the identity map) such that

$$(4.2) \quad \sum_{(i, \sigma_j) \in I_1} \sigma_j(\lambda_i) \sigma_j(\alpha_i)^n = 0 = \alpha_1^n \left(\sum_{(i, \sigma_j) \in I_1} \sigma_j(\lambda_i) \zeta_h^{(w_{ij} - w_{11})n} \right)$$

for all but finitely many $n \in \mathfrak{S}$.

We first prove that $I_1 = \mathcal{P} \times H$ for some subgroup $H \subseteq G$ and $\mathcal{P} = \{i : (i, \sigma) \in I_1\} \subseteq \{1, \dots, k\}$.

Note that if $i \in \mathcal{P}$, then there exists $\sigma \in G$ such that $(i, \sigma) \in I_1$. Therefore, for each $i \in \mathcal{P}$, we choose $\tau_i \in G$ such that $\alpha_1 \sim \tau_i(\alpha_i)$ and hence let $H_i := \{\sigma \in G \mid \tau_i(\alpha_i) \sim \sigma(\tau_i(\alpha_i))\}$. Then H_i is a subgroup of G by Proposition 3.2.

Note that $H_i = H_j$ for any $i, j \in \mathcal{P}$. For, if $\sigma \in H_i$, then $\alpha_1 \sim \tau_i(\alpha_i) \sim \sigma(\tau_i(\alpha_i))$. Since $(i, \tau_i), (i, \tau_j) \in I_1$, we see that $\tau_i(\alpha_i) \sim \tau_j(\alpha_j)$. Therefore, by acting σ on this equivalence, we get

$$\tau_j(\alpha_j) \sim \tau_i(\alpha_i) \sim \sigma(\tau_i(\alpha_i)) \sim \sigma(\tau_j(\alpha_j))$$

and hence $H_i \subseteq H_j$. Similarly, we get $H_j \subseteq H_i$. Since $(1, \sigma_1) \in H_1$, by taking $H = H_1$, we get $I_1 = \mathcal{P} \times H$.

We now rewrite (4.2) as

$$\sum_{i \in \mathcal{P}} \sum_{\sigma \in H} \sigma(\tau_i(\lambda_i)) \sigma(\tau_i(\alpha_i^n)) = \sum_{\sigma \in H} \sigma \left(\sum_{i \in \mathcal{P}} \tau_i(\lambda_i \alpha_i^n) \right) = 0$$

for all but finitely many values of $n \in \mathfrak{S}$. In particular, there exists a non-negative integer $a < h$ such that there are infinitely many $n \equiv a \pmod{h}$ with $n \in \mathfrak{S}$. For any such $n \equiv a \pmod{h}$ in \mathfrak{S} , by combining (4.2) and the above equality, we get

$$\sum_{\sigma \in H} \sum_{i \in \mathcal{P}} \sigma[\tau_i(\lambda_i) (\tau_i(\alpha_i))^a] = 0.$$

Since $\text{Tr}_{K/\mathbb{Q}}$ is invariant under the Galois action, by applying the trace operator $\text{Tr}_{K/\mathbb{Q}}$, we get

$$\sum_{i \in \mathcal{P}} \text{Tr}_{K/\mathbb{Q}}(\lambda_i \alpha_i^a) = 0.$$

Since $\lambda_i, \alpha_i \in L$, this proves the theorem. \square

Proof of Theorem 2.6. Let K be the number field that is obtained by adjoining all the zeros and poles of $f_i(X)$'s and λ_j 's with \mathbb{Q} . For each i , we write $f_i(X) = \frac{p_i(X)}{q_i(X)}$ for some coprime polynomials $p_i(X), q_i(X) \in \mathcal{O}_K[X]$. Let h be the order of the torsion subgroup of K^\times .

Since $(p_i(X), q_i(X)) = 1$ in $K[X]$ for all i , there exist polynomials $r_i(X), s_i(X) \in \mathcal{O}_K[X]$ and $\beta_i \neq 0 \in \mathcal{O}_K$ such that

$$r_i(X)p_i(X) + s_i(X)q_i(X) = \beta_i,$$

and hence $r_i(X)f_i(X) + s_i(X) = \frac{\beta_i}{q_i(X)}$. In order to prove $f_i(X) \in \mathcal{O}_K[X]$, first, we prove that $q_i(X)$ is a constant polynomial in $K[X]$ for each i . To do this, we need to understand the conjugate polynomials of $q_i(X)$. We shall define the following.

For any number field L , we let

$$\mathfrak{A}_L := \mathcal{O}_L \setminus \{ \text{Zeroes of } p_i(X), q_i(X), (q_j(X)p_i(X))^h - (q_i(X)p_j(X))^h \text{ for } 1 \leq i < j \leq k \}.$$

Note that this set is the complement of a finite set because $f_i(X)/f_j(X)$ is not constant, and also we are removing the solutions of the equation $f_i(X)/f_j(X) = \zeta_h^a$ for every $i \neq j$ and for some integer a .

By the definition of \mathfrak{A}_K , we see that for any $\gamma \in \mathfrak{A}_K$ (or \mathfrak{A}_L), the tuple $(f_1(\gamma), \dots, f_k(\gamma))$ is a non-degenerate tuple. Therefore, by Theorem 2.1, $f_i(\gamma) \in \mathcal{O}_K$ (or \mathcal{O}_L) for all i . Hence, the value $\beta_i/q_i(\gamma) \in \mathcal{O}_K$ for each $\gamma \in \mathfrak{A}_K$. Note also that \mathfrak{A}_K contains all but finitely many integers in it.

If possible, for some i , we suppose $q_i(X)$ is a non-constant polynomial. Let L be the Galois closure of $\mathbb{Q}(\beta_i, \text{coefficients of } q_i(X))$. Then the polynomial $Q_i(X) =$

$$\prod_{\sigma \in \text{Gal}(L(X)/\mathbb{Q}(X))} \sigma(q_i(X))$$

is also a non-constant polynomial in $\mathbb{Q}[X]$. Consider \mathfrak{A}_L

and conclude that $\beta_i/q_i(\gamma) \in \mathcal{O}_L$ for each $\gamma \in \mathfrak{A}_L$. Therefore, $N_{L/\mathbb{Q}}(\beta_i/q_i(\gamma)) \in \mathbb{Z}$ for every $\gamma \in \mathfrak{A}_L$. Note that for all but finitely many $n \in \mathbb{Z}$ lie in \mathfrak{A}_L and hence we have $N_{L/\mathbb{Q}}(\beta_i/q_i(n)) = N_{L/\mathbb{Q}}(\beta_i)/Q_i(n) \in \mathbb{Z}$. Since $|Q_i(n)| \rightarrow \infty$ as $|n| \rightarrow \infty$, we obtain $|N_{L/\mathbb{Q}}(\beta_i)/Q_i(n)| \rightarrow 0$ as $n \rightarrow \infty$. However, this is a sequence of integers and therefore we conclude that $N_{L/\mathbb{Q}}(\beta_i) = 0$. This implies $\beta_i = 0$, which is a contradiction. Hence $q_i(X)$ must be constant and hence $f_i(X) \in K[X]$ for each i .

We now proceed to show that $f_i(X) \in \mathcal{O}_K[X]$. Let $f_i(X) = \frac{p_i(X)}{\gamma_i} = \frac{1}{\gamma_i} \sum_{j=0}^d b_j X^j$

for some $\gamma_i, b_j \in \mathcal{O}_K$. Suppose there exists a prime ideal \mathfrak{P} in \mathcal{O}_K such that $v_{\mathfrak{P}}(b_j/\gamma_i) < 0$ for some j . We choose a number field L containing K having a prime ideal \mathfrak{Q} in \mathcal{O}_L lying above \mathfrak{P} such that the ramification index (say e) is greater than $2d$. Note that such L can be chosen by adjoining the appropriate root of unity to K to get the desired. Therefore, for any $\delta \in K^\times$, we have $v_{\mathfrak{Q}}(\delta) = e v_{\mathfrak{P}}(\delta)$. By the choice of e , we conclude that

$$(4.3) \quad e \min_{0 \leq r \leq d} v_{\mathfrak{P}}(b_r/\gamma_i) + d < 0.$$

Now since \mathfrak{A}_L is the complement of a finite set, we choose $\alpha \in \mathfrak{A}_L$ such that $v_{\mathfrak{Q}}(\alpha) = 1$. Then $v_{\mathfrak{Q}}(f_i(\alpha)) = \min_r v_{\mathfrak{Q}}\left(\frac{b_r}{\gamma_i} + r\right)$ because for any two distinct non-negative numbers $r, s \leq d$, due to (4.3), we have, $v_{\mathfrak{Q}}\left(\frac{b_s \alpha^s}{\gamma_i}\right) \neq v_{\mathfrak{Q}}\left(\frac{b_r \alpha^r}{\gamma_i}\right)$, as

$e > 2d$ (If they are equal, we take absolute values and consider their difference to get a contradiction). Therefore,

$$v_{\Omega}(f_i(\alpha)) \leq \min_r e \left[v_{\mathfrak{P}} \left(\frac{b_r}{\gamma_i} \right) \right] + d < 0,$$

a contradiction to the fact that $f_i(\alpha) \in \mathcal{O}_L$ as $\alpha \in \mathfrak{A}_L$. □

5. PROOFS OF THEOREMS 2.7 AND 2.8

Proof of Theorem 2.7. By Lemma 3.1, it is enough to prove the assertion for \mathbb{Q}_p for every prime number p .

Let p be a given prime number. Assume that α_1 is not integral over \mathbb{Z}_p and K be the Galois closure of $\mathbb{Q}_p(\alpha_1)$. All the other Galois conjugates of α_1 are $\alpha_2, \dots, \alpha_k$ for some integer k . It is also enough to prove the case when $b_3 = b_4 = \dots = b_k = 0$ and the proof for the general case follows verbatim.

Assume that α_1 is not integral over \mathbb{Z}_p (and so is α_2). For simplicity, we write $\alpha_1 = \alpha$ and $\alpha_2 = \beta$. Then α^{-1} and β^{-1} are integral over \mathbb{Z}_p and let the characteristic polynomial $f_{K|\mathbb{Q}_p}(x) := f(x)$ of α^{-1} satisfies the assertion in Lemma 3.2. Since β is a Galois conjugate of α , we see that the characteristic polynomial of β^{-1} is $f(x)$ itself. Write the unique maximal ideal $p\mathbb{Z}_p$ of \mathbb{Z}_p by \mathfrak{P} .

If $f(x) = x^d + a_{d-1}x^{d-1} + \dots + a_0$, then $f(\alpha^{-1}) = 0$ and $f(\beta^{-1}) = 0$. Hence we get

$$(5.1) \quad a_0 + a_1\alpha^{-1} + \dots + a_{d-1}\alpha^{-d+1} + \alpha^{-d} = 0 = a_0 + a_1\beta^{-1} + \dots + a_{d-1}\beta^{-d}.$$

Then, for any integer $\ell \geq 0$, multiplying by $\alpha^{d+\ell}$ both sides of (5.1), we get

$$(5.2) \quad \alpha^\ell = -a_{d-1}\alpha^{\ell+1} - \dots - a_0\alpha^{d+\ell} \text{ with } a_i \in \mathfrak{P}$$

and

$$(5.3) \quad \beta^\ell = -a_{d-1}\beta^{\ell+1} - \dots - a_0\beta^{d+\ell} \text{ with } a_i \in \mathfrak{P}$$

by Lemma 3.2. Now, for any integer $\ell \geq 0$, let M_ℓ be a \mathbb{Z}_p -submodule of K spanned by $b_1\alpha^{\ell+1} + b_2\beta^{\ell+1}, \dots, b_1\alpha^{\ell+d} + b_2\beta^{\ell+d}$. By Lemma 3.3, it is clear that M_ℓ is a non-zero \mathbb{Z}_p -submodule of K . Hence, by (5.2) and (5.3), we have

$$(5.4) \quad b_1\alpha^\ell + b_2\beta^\ell \in \mathfrak{P}M_\ell \text{ for any integer } \ell \geq 0.$$

Note that for any non-negative integers ℓ_1 and ℓ_2 , we have

$$(5.5) \quad M_{\ell_1} \subset M_{\ell_2} \text{ whenever } \ell_1 < \ell_2.$$

It is enough to prove that for $\ell_1 = \ell, \ell_2 = \ell + 1$, we have $b_1\alpha^{\ell+1} + b_2\beta^{\ell+1} \in M_{\ell_2}$ (and then inductively we can get the general assertion). Since $f(\alpha^{-1}) = 0 = f(\beta^{-1})$, multiplying by $\alpha^{\ell+1+d}$ on both sides, similarly, by $\beta^{\ell+1+d}$, we get $b_1\alpha^{\ell+1} + b_2\beta^{\ell+1} \in M_{\ell_2}$, as desired.

Now we claim that for any integer $\ell \geq 0$ and any integer $m \geq 0$, we have

$$(5.6) \quad b_1\alpha^\ell + b_2\beta^\ell \in \mathfrak{P}^{m+1}M_{\ell+dm}.$$

Let ℓ be any non-negative integer and $m = 0$. Then (5.6) is true by (5.4). Hence, we shall assume that (5.6) holds true for ℓ and for some integer $m \geq 1$. That is, we have $b_1\alpha^\ell + b_2\beta^\ell \in \mathfrak{P}^{m+1}M_{\ell+dm}$ and we prove (5.6) holds true for ℓ and $m + 1$. For

any integer i with $\ell + dm + 1 \leq i \leq \ell + d(m + 1)$, we have $b_1\alpha^i + b_2\beta^i \in M_{\ell+dm}$. By (5.4) and (5.5), we get

$$b_1\alpha^i + b_2\beta^i \in \mathfrak{P}M_i \subset \mathfrak{P}M_{\ell+d(m+1)} \text{ for all integers } i \text{ with } \ell + dm + 1 \leq i \leq \ell + d(m + 1)$$

and hence we get

$$(5.7) \quad M_{\ell+dm} \subset \mathfrak{P}M_{\ell+d(m+1)}.$$

Since, by the induction hypothesis, we have $b_1\alpha^\ell + b_2\beta^\ell \in \mathfrak{P}^{m+1}M_{\ell+dm}$, by (5.7), we arrive at

$$b_1\alpha^\ell + b_2\beta^\ell \in \mathfrak{P}^{m+2}M_{\ell+d(m+1)}$$

as desired.

Now to finish the proof, we take $\ell = 0$ and $m = v_p((b_1 + b_2)d)$. By hypothesis, we know that $\text{Tr}_{K|\mathbb{Q}_p}(M_{a-d}) \subset \mathbb{Z}_p$ for all integers $a \leq [d \log_2((b_1 + b_2)d)] + 1$. Since $dm < d \log_2((b_1 + b_2)d) + 1$, we get $\text{Tr}_{K|\mathbb{Q}_p}(M_{dm}) \subset \mathbb{Z}_p$. Therefore, since $b_1 + b_2 = b_1\alpha^0 + b_2\beta^0 \in \mathfrak{P}^{m+1}M_{dm}$, we see that $(b_1 + b_2)d = \text{Tr}_{K|\mathbb{Q}_p}(b_1\alpha^0 + b_2\beta^0) \in \text{Tr}_{K|\mathbb{Q}_p}(\mathfrak{P}^{m+1}M_{dm}) \subset \mathbb{Z}_p$. Therefore, we get $(b_1 + b_2)d \in \mathfrak{P}^{m+1}$ which implies that the power of p dividing $(b_1 + b_2)d$ is at least $m + 1$, a contradiction. Hence the theorem. \square

Proof of Theorem 2.8. Given that

$$C = 2 + \left\lceil \frac{1}{2} \max_{\mathfrak{P}} v_{\mathfrak{P}}(\lambda_1\lambda_2(\alpha_1 - \alpha_2)^2) \right\rceil + \max_{\mathfrak{P}} (|v_{\mathfrak{P}}(\lambda_1\lambda_2)|)$$

where \mathfrak{P} runs through all the prime ideals in \mathcal{O}_K and $\lambda_1\alpha_1^n + \lambda_2\alpha_2^n \in \mathcal{O}_K$ for all $1 \leq n \leq C$.

If possible, we assume that α_1 is not an algebraic integer. Then there exists a prime ideal \mathfrak{P} of \mathcal{O}_K such that $v_{\mathfrak{P}}(\alpha_1) < 0$.

We first claim that $v_{\mathfrak{P}}(\alpha_1) = v_{\mathfrak{P}}(\alpha_2)$. To show this, for each $i > |v_{\mathfrak{P}}(\lambda_1)|$, we have that $v_{\mathfrak{P}}(\lambda_1\alpha_1^i) = v_{\mathfrak{P}}(\lambda_1) + iv_{\mathfrak{P}}(\alpha_1) \leq v_{\mathfrak{P}}(\lambda_1) - i < 0$. Using the fact that $v_{\mathfrak{P}}(x + y) = \min\{v_{\mathfrak{P}}(x), v_{\mathfrak{P}}(y)\}$, when $v_{\mathfrak{P}}(x) \neq v_{\mathfrak{P}}(y)$, and that $v_{\mathfrak{P}}(\lambda_1\alpha_1^i + \lambda_2\alpha_2^i) \geq 0$ as $1 \leq i \leq C$, we conclude that

$$v_{\mathfrak{P}}(\lambda_2\alpha_2^i) = v_{\mathfrak{P}}(\lambda_1\alpha_1^i) \implies i|v_{\mathfrak{P}}(\alpha_1) - v_{\mathfrak{P}}(\alpha_2)| \leq |v_{\mathfrak{P}}(\lambda_1\lambda_2)|$$

holds for each $i > |v_{\mathfrak{P}}(\lambda_1)|$. Now, by choosing $i > |v_{\mathfrak{P}}(\lambda_1\lambda_2)|$, we conclude that $v_{\mathfrak{P}}(\alpha_1) = v_{\mathfrak{P}}(\alpha_2)$ as $v_{\mathfrak{P}}$ takes integer values. In particular, $v_{\mathfrak{P}}(\alpha_2) < 0$.

We note that for each $n \geq 1$,

$$(5.8) \quad \begin{aligned} V_n &:= [\lambda_1\alpha_1^n + \lambda_2\alpha_2^n][\lambda_1\alpha_1^{n+2} + \lambda_2\alpha_2^{n+2}] - [\lambda_1\alpha_1^{n+1} + \lambda_2\alpha_2^{n+1}]^2 \\ &= \lambda_1\lambda_2\alpha_1^n\alpha_2^n[\alpha_1 - \alpha_2]^2 = V_1[\alpha_1\alpha_2]^{n-1}. \end{aligned}$$

For $n > \max\{(v_{\mathfrak{P}}(V_1))/2 + 1, |v_{\mathfrak{P}}(\lambda_1\lambda_2)|\}$, we have

$$v_{\mathfrak{P}}(V_n) = v_{\mathfrak{P}}(V_1) + (n - 1)v_{\mathfrak{P}}(\alpha_1\alpha_2) \leq v_{\mathfrak{P}}(V_1) - 2(n - 1) < 0.$$

In particular, $v_{\mathfrak{P}}(V_C) < 0$, as $C > \max\{(v_{\mathfrak{P}}(V_1))/2 + 1, |v_{\mathfrak{P}}(\lambda_1\lambda_2)|\}$. This is not possible, because one notes that $V_n \in \mathcal{O}_K$ for all $n \leq C$ and in particular, $V_C \in \mathcal{O}_K$. Therefore $\alpha_1 \in \mathcal{O}_K$. Similarly, we can prove $\alpha_2 \in \mathcal{O}_K$. \square

Remark 5.1. One may try to generalise this argument for $k \geq 3$. There are issues with the valuation argument for more than 2 variables. We may use Hankel determinants of matrices with entries consisting only of $\lambda_1 \alpha_1^i + \cdots + \lambda_k \alpha_k^i$ to arrive at an equation very similar to (5.8). Proceeding in the same manner from there, one may obtain that $\prod_{i=1}^k \alpha_i \in \mathcal{O}_K$. However, it is not possible to obtain a bound purely depending on $\lambda_1 \alpha_1^i + \cdots + \lambda_k \alpha_k^i, \lambda_i$, by induction for the following reason: When we try to do induction over k , we know the values $\lambda_1 \alpha_1^i + \cdots + \lambda_k \alpha_k^i$ only for k terms and do not know for $k - 1$ terms. The process will follow through but we won't be able to determine the constant C , if we assume that $|\alpha_i|_{\mathfrak{p}} \leq 1$ for some i by this method.

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